1

Spaces

1.1 Linear Spaces and Linear Operators

As we have said in the preface, the concept of a *norm* is defined on linear spaces, and one of the important class of functions between normed linear spaces is the class of all linear operators.

1.1.1 Linear spaces

Definition 1.1.1 A **linear space** over the field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$, consists of a non-empty set X together with two operations,

$$(x, y) \mapsto x + y \qquad \text{(vector additon)},$$

$$(\alpha, x) \mapsto \alpha x \qquad \text{(scalar multiplication)},$$

defined on $X \times X$ and $\mathbb{K} \times X$, respectively, with values in X which satisfy the following conditions:

1. $x + y = y + x \quad \forall\, x, y \in X,$
2. $(x + y) + z = x + (y + z) \quad \forall\, x, y, z \in X,$
3. $\exists\, \theta \in X$ such that $\quad x + \theta = x \quad \forall\, x \in X,$
4. $\forall\, x \in X,\ \exists\, \tilde{x} \in X$ such that $\quad x + \tilde{x} = \theta.$
5. $\alpha(x + y) = \alpha x + \alpha y \quad \forall\, x, y \in X \quad \forall\, \alpha \in \mathbb{K},$
6. $(\alpha + \beta)x = \alpha x + \beta x \quad \forall\, x \in X$ and $\quad \forall\, \alpha, \beta \in \mathbb{K},$
7. $(\alpha\beta)x = \alpha(\beta x) \quad \forall\, x \in X$and $\quad \forall\, \alpha, \beta \in \mathbb{K},$
8. $1x = x \quad \forall\, x \in X.$

$\Diamond$

A few terminologies:

- A linear space is also called a *vector space*.
- Elements of a vector space are called *vectors*.
- Elements of the field $\mathbb{K}$ are called *scalars*.

Observe:

- The zero vector θ in condition 3 in Definition 1.1.1 is unique (Exercise). It is called the *zero vector*, and it is usually denoted by 0.
- The element $\tilde{x}$ in condition 4 in Definition 1.1.1is unique (Exercise), and it is usually denoted by $-x$.
- Condition 8 in Definition 1.1.1 enables us to have the following cancellation rule:
$$\alpha, \beta \in \mathbb{K}, \quad \theta \neq x \in X, \quad \alpha x = \beta x \Longrightarrow \alpha = \beta.$$
- By condition 2 in Definition 1.1.1, we can define sum of a finite number of elements $x_1, \ldots, x_n$ in a vector space as follows as
$$x_1 + \ldots + x_n,$$
which may be understood as the element y_n where $y_1, \ldots, y_n$ are defined iteratively as $y_1 = x_1$ and for $k = 2, \ldots, n$,
$$y_k = y_{k-1} + x_k.$$

Definition 1.1.2 Let X be a linear space.

1. A subset X_0 of X is said to be a **subspace** of X if X_0 itself is a linear space with respect to the operations of addition and scalar multiplication for X.
2. A subspace X_0 of X which is a proper subset of X is called a **proper subspace** of X.

$\Diamond$

We observe the following:

- A subset X_0 of a linear space X is a subspace if and only if for every $x, y \in X_0$ and $\alpha \in \mathbb{K}$,
$$x + y \in X_0 \quad \text{and} \quad \alpha x \in X_0.$$
- If X_1 and X_2 are subspaces of a vector space X, then $X_1 \cap X_2$ is a subspace of X, whereas $X_1 \cup X_2$ need not be a subspace.

In the following we list some of the standard examples of linear spaces that one studies in linear algebra:

Example 1.1.1 1. The space $\mathbb{K}^n$ of all n-tuples of numbers in $\mathbb{K}$, that is,
$$\mathbb{K}^n = \{(\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{K},\ i = 1, \ldots, n\}$$
with *coordinatewise* addition and scalar multiplication: If
$$x = (\alpha_1, \ldots, \alpha_n), \quad y = (\beta_1, \ldots, \beta_n), \quad \alpha \in \mathbb{K},$$
then $x + y$ and αx are defined by
$$x + y := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \quad \alpha x := (\alpha\alpha_1, \ldots, \alpha\alpha_n).$$

2. The space $\mathcal{P}_n$ of all polynomials of degree atmost n with coefficients from $\mathbb{K}$, that is,
$$p(t) \in \mathcal{P}_n \iff p(t) = a_0 + a_1 t + \ldots + a_n t^n$$
for some $a_0, a_1, \ldots, a_n$ in $\mathbb{K}$. Here addition and scalar multiplication are defined as follows: Let
$$p(t) = a_0 + a_1 t + \ldots + a_n t^n, \quad q(t) = b_0 + b_1 t + \ldots + b_n t^n$$
and $\alpha \in \mathbb{K}$. Then
$$p(t) + q(t) := (a_0 + b_0) + (a_1 + b_1)t + \ldots + (a_n + b_n)t^n,$$
$$\alpha p(t) := \alpha a_0 + \alpha a_1 t + \ldots + \alpha a_n t^n.$$

3. The space $\mathcal{P}$ of all polynomials with coefficients from $\mathbb{K}$, that is,
$$\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n.$$
Here addition and scalar multiplication are defined as follows: Let $p(t)$ and $q(t)$ be in $\mathcal{P}$ and $\alpha \in \mathbb{K}$. Then there exists $n \in \mathbb{N}$ such that both $p(t)$ and $q(t)$ belong to $\mathcal{P}_n$. Then $p(t) + q(t)$ and $\alpha p(t)$ are defined as in $\mathcal{P}_n$.

4. Let Ω be any non-empty set. Then $\mathcal{F}(\Omega)$ is the space of all $\mathbb{K}$-valued functions defined on Ω with addition and scalar multiplication defined pointwise, that is, for $f, g \in \mathcal{F}(\Omega)$ and $\alpha \in \mathbb{K}$,
$$(f+g)(t) = f(t) + g(t), \qquad t \in \Omega,$$
$$(\alpha f)(t) = \alpha f(t), \qquad t \in \Omega.$$
We see that the set $B(\Omega)$ of all $\mathbb{K}$-valued bounded functions on Ω is a subspace of $\mathcal{F}(\Omega)$.

5. Let Ω be a metric space or, more generally, a topological space. Then $C(\Omega)$, the space of all $\mathbb{K}$-valued continuous functions defined on Ω, is a subspace of $\mathcal{F}(\Omega)$. The subspace $C(\Omega) \cap B(\Omega)$ will be denoted by $C_b(\Omega)$.

6. For an interval J, let $\mathcal{P}_n(J)$ be the set of all elements in $\mathcal{P}_n$ considered as $\mathbb{K}$-valued functions defined on J, and
$$\mathcal{P}(J) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(J).$$
Thus, $p \in \mathcal{P}_n(J)$ if and only if there exists $a_0, a_1, \ldots, a_n$ in $\mathbb{K}$ such that
$$p(t) = a_0 + a_1 t + \ldots + a_n t^n \quad \forall\, t \in J,$$
and $p \in \mathcal{P}(J)$ if and only if there exists $n \in \mathbb{N}$ such that $p \in \mathcal{P}_n(J)$. Thus, for an interval J,
   - $\mathcal{P}_n(J)$ is a subspace of $\mathcal{P}(J)$ and
   - $\mathcal{P}(J)$ is a subspace of $C(J)$.

7. If f is a complex valued function defined on $[a, b]$, then we say that f is Riemann integrable if its real and imaginary parts, namely, $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, respectively, are Riemann integrable, and in that case we define
$$\int_a^b f(t)\, dt = \int_a^b \mathrm{Re}(f)(t) dt + i \int_a^b \mathrm{Im}(f)(t) dt.$$
Here, i denotes the complex number whose square is -1. Let $\mathcal{R}[a, b]$ be the set of all $\mathbb{K}$- valued Riemann integrable functions on $[a, b]$. Then $\mathcal{R}[a, b]$ is a subspace of $\mathcal{F}([a, b])$.

8. As a special case of $\mathcal{F}(\Omega)$, taking $\Omega = \mathbb{N}$, we obtain the space of all scalar sequences. We may recall from real analysis that a sequence of scalars is often written as (α_n) or $(\alpha_1, \alpha_2, \ldots)$ whenever $\alpha_n = x(n)$, $n \in \mathbb{N}$ for some $x \in \mathcal{F}(\mathbb{N})$. Thus, we identify the space

$$X = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{K} \text{ for } i \in \mathbb{N}\}$$

of all infinite-tuples with the space $\mathcal{F}(\mathbb{N})$. In the sequel we shall consider many subspaces of the above sequence space by imposing conditions on its elements. Some of the standard subspaces of $\mathcal{F}(\mathbb{N})$ are the following:

(a) The space of all bounded sequences, denoted by ℓ^∞.

(b) The space of all convergent sequences, denoted by c.

(c) The space all convergent sequences with limit 0, denoted by c_0.

(d) The space of all sequences having only a finite number of non-zero entries, denoted by c_{00}.

We observe:

- c_{00} is a proper subspace of c_0,
- c_0 is a proper subspace of c,
- c is a proper subspace of ℓ^∞.

Also, note that, with respect to the usual metric on $\mathbb{N}$,

$$C(\mathbb{N}) = \mathcal{F}(\mathbb{N}), \quad C_b(\mathbb{N}) = \ell^\infty = B(\mathbb{N}).$$

$\square$

Notation 1.1.1 The space $\mathbb{K}^n$ is also realized as $\mathcal{F}(\Omega)$ with $\Omega = \{1, \ldots, n\}$. When $\mathbb{K}^n$ is in this *avatar*, the coordinates of an element x in $\mathbb{K}^n$ are written as $x(1), \ldots, x(n)$. Thus, $x = (x(1), \ldots, x(n))$. Also, the n^{th} coordinates of an element $x \in \mathcal{F}(\mathbb{N})$ is written as $x(n)$ so that we may also write x as $(x(1), x(2), \ldots)$. $\Diamond$

1.1.2 Linear independence, basis and dimension

Definition 1.1.3 Let E be a subset of a linear space X.

1. E is said to be **linearly dependent** if there exist $u_1, \dots, u_n$ in E and $\alpha_1, \dots, \alpha_n$ in $\mathbb{K}$, with atleast one of them nonzero, such that
$$\alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

2. E is said to be **linearly independent** if it is not linearly dependent.

3. An element $x \in X$ is a **finite linear combination** of members of E if there exist $u_1, \dots, u_n$ in E and $\alpha_1, \dots, \alpha_n$ in $\mathbb{K}$ such that
$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n$$
and in that case x is also called a **linear combination** of $u_1, \dots, u_n$.

4. **Span** of E, written $\text{span}\,(E)$, is the set of all finite linear combinations of members of E.

5. E is a **basis** of X if it is linearly independent and $\text{span}\,(E) = X$.

$\Diamond$

Let X be a linear space. We observe:

- A subset E of X is linearly independent if and only if either $E = \varnothing$ or for any finite number of vectors $u_1, \dots, u_n$ in E,
$$\alpha_1 u_1 + \cdots + \alpha_n u_n = 0 \quad \Longrightarrow \quad \alpha_1 = 0, \dots, \alpha_n = 0.$$
- For any subset E of X, $\text{span}\,(E)$ is a subspace of X.
- If E is linearly independent in X, then E is a basis for $\text{span}\,(E)$.

It can be shown that (see Nair [5])

- If X has a basis consisting of n elements, then any subset of X containing more than n elements is linearly dependent.

Consequently:

- If X has a finite basis, then any two bases of X have the same number of elements.

Definition 1.1.4 Let X be a vector space.

1. X is said to be a **finite dimensional vector space** if there is a finite basis for X, and in that case the number of elements in a basis is called the **dimension** of X, and we write this number as $\dim(X)$.

2. X is said to be an **infinite dimensional vector space** if it is not a finite dimensional vector space, and in that case we say that the dimension of X is infinity and we write this fact as $\dim(X) = \infty$.

$\Diamond$

Using *Zorn's lemma*, it can be shown that (see Nair [5])

- Every vector space has a basis.

Observe:

- If a vector space contains an infinite linearly independent set, then it is infinite dimensional.

Suppose $E = \{u_1, \ldots, u_n\}$ is a basis of a (finite dimensional) vector space X. If φ is a bijective function from $\{1, \ldots, n\}$ to itself which is not the identity function and if $\alpha_1, \ldots, \alpha_n$ are in $\mathbb{K}$, then we know that the linear combinations

$$\alpha_1 u_1 + \ldots \alpha_n u_n \quad \text{and} \quad \alpha_1 u_{\varphi(1)} + \ldots + \alpha_n u_{\varphi(n)}$$

are different. Thus, the order in which the basis elements appear in a linear combination matters.

Definition 1.1.5 Let X be a vector space with a countable basis E. The set E with its elements arranged in a particular order is called an **ordered basis**. $\Diamond$

Thus, if X is finite dimensional and if $E = \{u_1, \ldots, u_n\}$ is a basis of X, then for each bijective function φ from $\{1, \ldots, n\}$ to itself, we obtain an ordered basis

$$E_\varphi = \{u_{\varphi(1)}, \ldots, u_{\varphi(n)}\}.$$

Similarly if X is infinite dimensional with a countably infinite basis $E = \{u_1, u_2, \ldots\}$, then for each bijective function φ from $\mathbb{N}$ to itself, we obtain an ordered basis

$$E_\varphi = \{u_{\varphi(1)}, u_{\varphi(2)}, \ldots\}.$$

Notation 1.1.2 In the sequel we shall use the following *Dirac function*: If S is a nonempty set, then for every $s \in S$, the function $e_s : S \to \mathbb{R}$ is defined by

$$e_s(t) = \delta_{st} := \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

$\Diamond$

Using the above notation, if $S = \{1, \ldots, n\}$, then for each $i \in S$, e_i can be identified with the element $(\delta_{i1}, \ldots, \delta_{in})$ in $\mathbb{K}^n$, and if $S = \mathbb{N}$, then e_i can be identified with the sequence $(\delta_{i1}, \delta_{i2}, \ldots)$.

Example 1.1.2 1. Let $e_i(j) = \delta_{ij}$ for $i, j = 1, \ldots, n$ and let $E = \{e_1, \ldots, e_n\}$. It can be verified that E is a basis of $\mathbb{K}^n$. This basis is called the *standard basis* of $\mathbb{K}^n$.

2. Let $e_i(j) = \delta_{ij}$ for $i, j \in \mathbb{N}$ and let $E = \{e_1, e_2, \ldots\}$. It can be verified that

   (a) E is a basis of c_{00},

   (b) E is linearly independent in c_0, but not a basis of c_0. For instance, a sequence in c_0 with infinite number of nonzero components cannot be written as a finite linear combination of members of E.

3. $\{1, t, \ldots, t^n\}$ is a basis of $\mathcal{P}_n$.

4. $\{1, t, t^2, \ldots\}$ is a basis of $\mathcal{P}$.

5. If $u_\lambda(t) = e^{\lambda t}$ for $t \in [a, b]$ and $\lambda \in \mathbb{R}$, then $\{u_\lambda : \lambda \in \mathbb{R}\}$ is linear independent in $C^k[a, b]$ for any $k \in \mathbb{N}$. Here, $C^k[a, b]$ is the vector space of all k times continuously differentiable functions on $[a, b]$.

$\square$

1.1.3 Linear transformation

We observe that if X is a finite dimensional linear space and $\{u_1, \ldots, u_n\}$ is an ordered basis of X, then the function

$$\sum_{i=1}^{n} \alpha_i u_i \mapsto (\alpha_1, \ldots, \alpha_n)$$

transfers the linear structure of X to that of $\mathbb{K}^n$ and vice versa. To make this more precise let us recall, from linear algebra, the definition of a linear operator.

Definition 1.1.6 Let X and Y be linear spaces. Then a function $T : X \to Y$ is said to be a **linear transformation** or a **linear operator** if

$$T(x + y) = T(x) + T(y), \qquad x, y \in X,$$

$$T(\alpha x) = \alpha T(x) \qquad x \in X, \alpha \in \mathbb{K}.$$

A linear operator with its codomain as the field of scalars is called a **linear functional**. ♢

Notation 1.1.3 Value of a linear operator $T : X \to Y$ at a point $x \in X$ is also denoted by Tx. Linear functionals are usually denoted by small letters such as f, g, etc. ♢

It is to be observed that the *null space* of T, namely,

$$N(T) = \{x \in X : Tx = 0\},$$

and the *range space* of T, namely,

$$R(T) = \{y \in Y : \exists\, x \in X \text{ such that } y = Tx\},$$

are subspaces of X and Y, respectively.

For a linear operator $T : X \to Y$, we observe:

- T is one-one or injective if and only if $N(T) = \{0\}$, and it is onto or surjective if and only if $R(T) = Y$.

- If T is injective, then its inverse from its range, namely the function $T^{-1} : R(T) \to X$, is also a linear operator.

Definition 1.1.7 Let X and Y be linear spaces.

1. The spaces X and Y are said to be **isomorphic** or **linearly isomorphic** if there exists a bijective linear operator from X to Y.

2. A bijective linear operator between X and Y is called a **linear isomorphism** between X and Y.

◊

Suppose $T : X \to Y$ be a linear operator between linear spaces X and Y. Then we observe the following:

- If T is injective and $u_1, \ldots, u_n$ are linearly independent in X, then $Tu_1, \ldots, Tu_n$ are linearly independent in Y.
- If $u_1, \ldots, u_n$ are in X such that $Tu_1, \ldots, Tu_n$ are linearly independent in Y, then $u_1, \ldots, u_n$ are linearly independent in X.
- If $\dim(X) = \dim(Y)$, then T is one-one if and only if T is onto.

To brush up our memory, in the following we present a few examples of linear operators and linear functionals:

Example 1.1.3 1. Let X be a finite dimensional linear space and let $E := \{u_1, \ldots, u_n\}$ be an ordered basis of X. For $x = \sum_{j=1}^n \alpha_j u_j \in X$, let

$$[x]_E := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Then the function $x \mapsto [x]_E$ defines a linear transformation from X to $\mathbb{K}^n$, which is bijective, so that it is a linear isomorphism between X and $\mathbb{K}^n$.

2. Let $A = (a_{ij})$ be an $m \times n$ matrix over $\mathbb{K}$. Let $T : \mathbb{K}^n \to \mathbb{K}^m$ be defined by

$$T(x) = A[x]_E, \qquad x \in \mathbb{K}^n,$$

where E is the standard basis of $\mathbb{K}^n$. Then T is a linear operator.

More generally we have the following:

3. Let X and Y be finite dimensional linear spaces, and $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be ordered basis of X and Y, respectively.

Let (a_{ij}) be an $m \times n$ matrix over $\mathbb{K}$. Corresponding to each $x = \sum_{j=1}^{n} \alpha_j u_j \in X$, let

$$T(x) = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}\alpha_j \Big) v_i.$$

Then $T : X \to Y$ is a linear operator. Here are a few special cases:

(a) If $m = n$ and (a_{ij}) is the identity matrix, then T is a linear isomorphism satisfying $T(u_i) = v_i$, $i = 1, \ldots, n$. Thus, any two vector spaces of the same dimension are linearly isomorphic.

(b) If $m = 1$, $Y = \mathbb{K}$ and $v_1 = 1$, then T is a linear functional given by

$$T(x) = a_1\alpha_1 + \ldots + a_n\alpha_n,$$

where $a_i := a_{i1}$.

$\square$

Let X and Y be finite dimensional linear spaces, and $T : X \to Y$ be a linear operator. Let $E := \{u_1, \ldots, u_n\}$ and $F := \{v_1, \ldots, v_m\}$ be ordered basis of X and Y, respectively. For each $j \in \{1, \ldots, n\}$, let a_{ij}, $i = 1, \ldots, m$ be such that

$$Tu_j = \sum_{i=1}^{m} a_{ij} v_i.$$

Then for every $x = \sum_{j=1}^{n} \alpha_j u_j \in X$,

$$Tx = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}\alpha_j \Big) v_i.$$

The $m \times n$ matrix (a_{ij}) is called the *matrix representation* of T with respect to the ordered bases E and F, and it is usually denoted by

$$[T]_{E,F}.$$

Using the notation introduced in Example 1.1.3, we see that

$$[Tx]_F = [T]_{E,F}[x]_E \quad \forall\, x \in X.$$

Example 1.1.4 Let $X = C[a,b]$.

1. The function $T : X \to X$ defined by
$$(Tx)(s) = \int_a^s x(t)dt, \quad x \in C[a,b],\, s \in [a,b],$$
is a linear operator.

2. The function $f : X \to \mathbb{K}$ defined by
$$f(x) = \int_a^b x(t)dt, \quad x \in C[a,b],$$
is a linear functional.

3. For each $\tau \in [a,b]$, the function $f_\tau : X \to \mathbb{K}$ defined by
$$f_\tau(x) = x(\tau), \quad x \in C[a,b],$$
is a linear functional.

$\square$

Example 1.1.5 Let $X = C^1[a,b]$ be the space of all $\mathbb{K}$-valued continuous functions which have continuous derivatives on $[a,b]$, where derivative at a and b are understood to be the left and the right derivative, respectively.

1. The function $T : X \to C[a,b]$ defined by
$$T(x) = x', \qquad x \in X,$$
is a linear operator.

2. For each $\tau \in [a,b]$, the function $f_\tau : X \to \mathbb{K}$ defined by
$$f_\tau(x) = x'(\tau), \qquad x \in X,$$
is a linear functional.

$\square$

1.2 Normed Linear Spaces

So far we have been discussing linear spaces and linear transformations. Now, we shall introduce an additional structure on a linear space which makes it a metric space, so that results from metric spaces can be used for analysis on linear spaces.

1.2.1 Norm on a linear space

Definition 1.2.1 Let X be a linear space (over $\mathbb{K}$). Then a **norm** on X is a function which assigns each $x \in X$ a unique non-negative real number, denoted by $\|x\|$, such that the following conditions are satisfied.

(i) For $x \in X$, $\|x\| = 0 \iff x = 0$.

(ii) For every x, y in X, $\|x + y\| \leq \|x\| + \|y\|$.

(iii) For every $x \in X$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha| \, \|x\|$.

A linear space together with a norm is called a **normed linear space** . $\Diamond$

Condition (i) is called the *positive definiteness* and condition (ii) is called the *triangle inequality.* The reason behind the nomenclature *triangle inequality* will be clear once we give the example of the *Euclidean norm* on the linear space $\mathbb{R}^2$

- If X_0 is a subspace of a normed linear space X with norm $\|\cdot\|$, then $\|\cdot\|$ is a norm on X_0 as well.

Theorem 1.2.1 *Suppose X is a normed linear space. Then the function $d : X \times X \to \mathbb{R}$ defined by*

$$d(x, y) = \|x - y\|, \qquad x, y \in X,$$

is a metric on X.

Proof. Let $x, y, z \in X$. From the definition of the norm, it follows that

$$\|x - y\| \geq 0,$$

$$\|x - y\| = 0 \iff x = y,$$

$$\|x - y\| = \|(x - z) + (z - y)\| \leq \|x - y\| + \|y - z\|.$$

Thus, d satisfies all the conditions for it to be a metric. ∎

The metric in the above theorem is called the metric **induced by the norm**.

- A bounded metric on a nonzero linear space does not induce a norm.

Throughout this book, we shall make use of concepts from metric space for a normed linear space, where the metric is understood to be the induced metric.

Let X be a normed linear space. Using the induced metric on X, we write a few definitions and results. More definitions and results will be introduced or recalled in the due course.

Definitions:

1. A sequence (x_n) in X *converges* to an element $x \in X$ if for every $\varepsilon > 0$, there exists a positive integer N such that
$$\|x_n - x\| < \varepsilon \qquad \forall\, n \geq N.$$

2. A sequence (x_n) in X is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists a positive integer N such that
$$\|x_n - x_m\| < \varepsilon \qquad \forall\, n, m \geq N.$$

3. A subset S of X is bounded if and only there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in S$.

4. A point $x_0 \in X$ is an *interior point* of a set $S \subseteq X$ if there exists $r > 0$ such that the *open ball*
$$B(x_0, r) := \{x \in X : \|x - x_0\| < r\}$$
is contained in S.

5. The set of all interior points of a set $S \subseteq X$ is called the *interior* of S, and it is denoted by S° or $\mathrm{int}(S)$.

6. A subset S of X is an open set if every point in S is an interior point of S.

7. A subset S of X is a closed set if it is compliment of an open set.

8. A point $x_0 \in X$ is a *limit point* of a set $S \subseteq X$ if for every $r > 0$, the open ball $B(x_0, r)$ contains a point of S and also a point of its compliment.

9. The *closure* of a set $S \subseteq X$ is the set S together with all its limit points, and it is denoted by $\overline{S}$ or $\mathrm{cl}\,(S)$.

10. If Y is also a normed linear space and $D \subseteq X$, then a function $F : D \to Y$ is continuous at a point $x_0 \in D$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$x \in D, \quad \|x - x_0\| < \delta \Longrightarrow \|F(x) - F(y)\| < \varepsilon.$$

Results:

1. A sequence (x_n) in X is a bounded sequence if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded.
2. Every convergent sequence is a Cauchy sequence.
3. Every Cauchy sequence in X is bounded.
4. Every Cauchy sequence having a convergent subsequence is also convergent.
5. A set S is closed if and only if for every sequence (x_n) in S, if (x_n) converges to some $x \in X$, then $x \in S$.
6. A set S is open if and only if $S^\circ = S$.
7. A set S is closed if and only if $\overline{S} = S$.
8. A set S is totally bounded if and only if every sequence in S has a Cauchy subsequence.
9. A set S is compact if and only if every sequence in S has a subsequence which converges to an element (point) in S.

Definition 1.2.2 A normed linear space is called a **Banach space** if it is complete, that is every Cauchy sequence in it converges with respect to the induced metric. $\Diamond$

1.2.2 Examples of norms

Assertions in the following examples for which proof is not given are to verified by the reader.

Example 1.2.1 For $x \in \mathbb{K}^n$, define
$$\|x\|_1 := \sum_{i=1}^{n} |x(i)|, \qquad \|x\|_\infty = \max\{|x(i)| : i = 1, \dots, n\},$$
Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{K}^n$. $\square$

Example 1.2.2 For $x \in C[a,b]$, let

$$\|x\|_1 := \int_a^b |x(t)|\,dt, \qquad \|x\|_\infty = \sup_{a\le t\le b} |x(t)|.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $C[a,b]$. □

For the next two examples we shall make use of the *Schwarz inequality* on $\mathbb{K}^n$ and on $C[a,b]$ as given in the following theorem.

Theorem 1.2.2 (Schwarz inequality)

(i) *For* $x, y \in \mathbb{K}^n$,

$$\sum_{i=1}^n |x(i)y(i)| \le \Big(\sum_{i=1}^n |x(i)|^2\Big)^{1/2}\Big(\sum_{i=1}^n |y(i)|^2\Big)^{1/2}.$$

(ii) *For* x *and* y *in* $C[a,b]$,

$$\int_a^b |x(t)y(t)|\,dt \le \Big(\int_a^b |x(t)|^2\Big)^{1/2}\Big(\int_a^b |y(t)|^2\Big)^{1/2}.$$

Proof. (i) For $x \in \mathbb{K}^n$, let

$$\|x\|_2 = \Big(\sum_{i=1}^n |x(i)|^2\Big)^{1/2}.$$

Clearly, the inequality holds if either $x = 0$ or $y = 0$. Hence, assume that both x and y are nonzero vectors, so that $\|x\|_2$ and $\|y\|_2$ are nonzero. Then, using the relation

$$ab \le \frac{a^2+b^2}{2}$$

for every $a, b \in \mathbb{R}$, we obtain

$$\frac{|x(i)|}{\|x\|_2}\,\frac{|y(i)|}{\|y\|_2} \le \frac{1}{2}\left(\frac{|x(i)|^2}{\|x\|_2^2} + \frac{|y(i)|^2}{\|y\|_2^2}\right),$$

for each $i = 1, \ldots, n$, so that taking sum, we have

$$\sum_{i=1}^n \frac{|x(i)|}{\|x\|_2}\,\frac{|y(i)|}{\|y\|_2} \le \frac{1}{2}\sum_{i=1}^n \left(\frac{|x(i)|^2}{\|x\|_2^2} + \frac{|y(i)|^2}{\|y\|_2^2}\right) = 1.$$

Thus, the inequality in (i) is proved.

(ii) For $x \in C[a,b]$, let

$$\|x\|_2 = \Big(\int_a^b |x(t)|^2\Big)^{1/2}.$$

As in (i), we have

$$\frac{|x(t)|}{\|x\|_2}\,\frac{|y(t)|}{\|y\|_2} \le \frac{1}{2}\left(\frac{|x(t)|^2}{\|x\|_2^2} + \frac{|y(t)|^2}{\|y\|_2^2}\right),$$

for each $t \in [a,b]$, so that taking integral, we have

$$\int_a^b \left(\frac{|x(t)|}{\|x\|_2}\,\frac{|y(t)|}{\|y\|_2}\right) dt \le \frac{1}{2}\int_a^b \left(\frac{|x(t)|^2}{\|x\|_2^2} + \frac{|y(t)|^2}{\|y\|_2^2}\right) dt = 1.$$

From this, we obtain the inequality in (ii). ∎

Example 1.2.3 For $x \in \mathbb{K}^n$, let

$$\|x\|_2 := \Big(\sum_{i=1}^n |x(i)|^2\Big)^{1/2}.$$

We show that $\|\cdot\|_2$ is a norm on $\mathbb{K}^n$. For this, first we observe that $\|x\|_2 \ge 0$ for all $x \in \mathbb{K}^n$. Further the conditions (i) and (iii) in the Definition 1.2.1 can be easily verified. Hence, we need to verify only the condition (ii). Note that

$$\sum_{i=1}^n |x(i) + y(i)|^2 \le \sum_{i=1}^n (|x(i)|^2 + |y(i)|^2 + 2|x(i)|\,|y(i)|).$$

By the Schwarz inequality (i) in Theorem 1.2.2,

$$\sum_{i=1}^n |x(i)|\,|y(i)| \le \|x\|_2\,\|y\|_2.$$

Hence, we obtain,

$$\|x+y\|_2^2 \le \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\,\|y\|_2 = (\|x\|_2 + \|y\|_2)^2$$

and consequently,

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2.$$

Thus, we have shown that $\|\cdot\|_2$ is also a norm on $\mathbb{K}^n$. □

The norm $\|\cdot\|_2$ on $\mathbb{K}^n$ defined in the above example is called the **Euclidean norm** on $\mathbb{K}^n$.

Example 1.2.4 For $x \in C[a,b]$, let

$$\|x\|_2 = \Big(\int_a^b |x(t)|^2\Big)^{1/2}.$$

Note that, for $x \in C[a,b]$, $\|x\|_2 \geq 0$ and $\|x\|_2 = 0$ if and only if $x = 0$. As in Example 1.2.3, replacing summation by integrals, we see that $x \mapsto \|x\|_2$ is also a norm on $C[a,b]$. □

One may wonder whether

$$\|x\|_p = \Big(\sum_{i=1}^{n} |x(i)|^p\Big)^{1/p}, \quad x \in \mathbb{K}^n,$$

defines a norm on $\mathbb{K}^n$. It does if and only if $1 \leq p < \infty$. To prove this we first establish an inequality called the *Hölder's inequality*, for which we shall make use of the following lemma.

Lemma 1.2.3 *Let p and q be positive real numbers satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for every positive real numbers a and b,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Recall from real analysis that a real valued function φ defined on an interval J is convex if for every $\alpha, \beta \in J$ and for $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$,

$$\varphi(\lambda\alpha + \mu\mu) \leq \lambda\varphi(\alpha) + \mu\varphi(\beta).$$

We know that the function $\varphi(t) := e^t$, $t > 0$, is a convex ([3], Proposition 4.3.1). Thus, for every $\alpha, \beta \in J$ and for $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$,

$$e^{\lambda\alpha+\mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta};$$

equivalently,

$$e^{\lambda\alpha} e^{\mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta};$$

Taking $\lambda = 1/p$, $\mu = 1/q$ and α and β such that $a = e^{\alpha/p}$, $b = e^{\beta/q}$, we obtain the required inequality. ∎

It is obvious that if p and q are positive real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$, then $1<p<\infty$ and $1<q<\infty$.

Theorem 1.2.4 (Hölder's inequality). *Let p and q be positive real numbers satisfying*

$$\frac{1}{p}+\frac{1}{q}=1.$$

For $x, y \in \mathbb{K}^n$,

$$\sum_{i=1}^{n}|x(i)y(i)| \leq \Big(\sum_{i=1}^{n}|x(i)|^p\Big)^{1/p}\Big(\sum_{i=1}^{n}|y(i)|^q\Big)^{1/q}.$$

Proof. For $x \in \mathbb{K}^n$, let

$$\|x\|_p = \Big(\sum_{i=1}^{n}|x(i)|^p\Big)^{1/p}$$

Note that

$$\|x\|_p = 0 \iff x = 0.$$

Thus, for $x, y \in \mathbb{K}^n$, if atleast one of x and y is 0, then the inequality holds trivially. So, we assume that both x and y are nonzero vectors. For each $i \in \{1,\ldots,n\}$, taking

$$a = \frac{|x(i)|}{\|x\|_p}, \quad b = \frac{|y(i)|}{\|y\|_q},$$

in Lemma 1.2.3, we have

$$\frac{|x(i)y(i)|}{\|x\|_p\|y\|_q} \leq \frac{|x(i)|^p}{p\|x\|_p^p} + \frac{|y(i)|^q}{q\|y\|_q^q}.$$

Taking sum, we obtain

$$\sum_{i=1}^{n}\frac{|x(i)y(i)|}{\|x\|_p\|y\|_q} \leq \sum_{i=1}^{n}\frac{|x(i)|^p}{p\|x\|_p^p} + \sum_{i=1}^{n}\frac{|y(i)|^q}{q\|y\|_q^q} = \frac{1}{p}+\frac{1}{q} = 1.$$

From this, the required inequality follows. ∎

Theorem 1.2.5 *For $1 \leq p < \infty$, let*

$$\|x\|_p = \Big(\sum_{i=1}^{n}|x(i)|^p\Big)^{1/p}, \quad x \in \mathbb{K}^n.$$

Then $x \mapsto \|x\|_p$ is a norm on $\mathbb{K}^n$.

Proof. The case of $p = 1$ has already been considered in Example 1.2.1. So, assume that $p > 1$ and let $x, y \in \mathbb{K}^n$. Note that

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^n |x(i)+y(i)|^p \\
&= \sum_{i=1}^n |x(i)+y(i)|\,|x(i)+y(i)|^{p-1} \\
&\le \sum_{i=1}^n |x(i)|\,|x(i)+y(i)|^{p-1} + \sum_{i=1}^n |y(i)|\,|x(i)+y(i)|^{p-1}.
\end{aligned}$$

Applying Hölder's inequality (Theorem 1.2.4), we have

$$\begin{aligned}
\sum_{i=1}^n |x(i)|\,|x(i)+y(i)|^{p-1} &\le \|x\|_p \Big(\sum_{i=1}^n |x(i)+y(i)|^{(p-1)q}\Big)^{1/q} \\
&= \|x\|_p \|x+y\|_p^{p/q}.
\end{aligned}$$

Similarly,

$$\sum_{i=1}^n |y(i)|\,|x(i)+y(i)|^{p-1} \le \|y\|_p \|x+y\|_p^{p/q}.$$

Hence,

$$\|x+y\|_p^p \le (\|x\|_p + \|y\|_p)\|x+y\|_p^{p/q}.$$

From this, we obtain

$$\|x+y\|_p \le \|x\|_p + \|y\|_p \qquad \forall\, x, y \in \mathbb{K}^n.$$

All other conditions for $\|\cdot\|_p$ to be a norm can be easily verified. ∎

Note that the norm $x \mapsto \|x\|_2$ on $\mathbb{K}^n$ is the Euclidean norm.

Remark 1.2.1 It is to be observed that for $n > 1$, if $0 < p < 1$, then

$$\|x\|_p = \Big(\sum_{i=1}^n |x(i)|^p\Big)^{1/p}, \quad x \in \mathbb{K}^n,$$

does not define a norm. To see this, it is enough to observe that

$$\|e_1 + e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p.$$

◊

Example 1.2.5 Let Ω be a nonempty set. Recall that $B(\Omega)$, the set of all $\mathbb{K}$-valued bounded functions on Ω, is a vector space with respect to the addition and scalar multiplication defined pointwise. It can be easily seen that

$$x \mapsto \|x\|_\infty := \sup_{t\in\Omega} |x(t)|$$

is a norm on $B(\Omega)$. In particular, taking $\Omega = \mathbb{N}$, then norm on $B(\mathbb{N})$ is given by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x(n)|, \quad x \in B(\mathbb{N}).$$

Since the space $C[a,b]$ and the space $\mathcal{R}[a,b]$ are subspaces of $B([a,b])$, $\|\cdot\|_\infty$ is a norm on these spaces as well. $\square$

Notation 1.2.1 The space $B(\Omega)$ is also denoted by $\ell^\infty(\Omega)$. The space $\ell^\infty(\mathbb{N})$ is usually denoted simply by ℓ^∞. $\Diamond$

Next we consider p-norms on $C[a,b]$. For that purpose, we require the Hölder's inequality on $C[a,b]$.

Theorem 1.2.6 (Hölder's inequality). *Let p and q be positive real numbers satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for $x, y \in C[a,b]$,

$$\int_a^b |x(t)y(t)|\, dt \le \Big(\int_a^b |x(t)|^p\Big)^{1/p} \Big(\int_a^b |y(t)|^q\Big)^{1/q}.$$

Proof. For $x \in C[a,b]$, let

$$\|x\|_p = \Big(\int_a^b |x(t)|^p\Big)^{1/p}$$

Let $x, y \in C[a,b]$. We have to prove that

$$\int_a^b |x(t)y(t)|\, dt \le \|x\|_p \|y\|_q.$$

This follows as in Theorem 1.2.4 by replacing sums by integrals. ∎

Remark 1.2.2 With the convention $1/\infty = 0$, we say that numbers p, q in $[1, \infty]$ are **conjugate exponents** if they satisfy

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and in that case we may say p **is a conjugate exponent to** q, and vice versa. Now, for any $x, y \in \mathbb{K}^n$, we have

$$\sum_{i=1}^{n} |x(i)y(i)| \leq \|x\|_1 \, \|y\|_\infty,$$

and for any $x, y \in C[a, b]$,

$$\int_a^b |x(t)y(t)|, dt \leq \|x\|_1 \, \|y\|_\infty.$$

Thus, for all $p, q \in [1, \infty]$ which are conjugates to each other, we have the **Hölder's inequalities**

$$\sum_{i=1}^{n} |x(i)y(i)| \leq \|x\|_p \, \|y\|_q \quad \forall\, x, y \in \mathbb{K}^n,$$

$$\int_a^b |x(t)y(t)| \, dt \leq \|x\|_p \, \|y\|_q \quad \forall\, x, y \in C[a, b].$$

$\Diamond$

Example 1.2.6 For $1 \leq p < \infty$, let

$$\ell^p = \Big\{ x \in \mathcal{F}(\mathbb{N}) : \sum_{i=1}^{\infty} |x(i)|^p \text{ converges} \Big\}.$$

For $p \in [1, \infty]$, define

$$\|x\|_p = \begin{cases} \Big(\sum_{i=1}^{\infty} |x(i)|^p \Big)^{1/p} & \text{if} \quad x \in \ell^p,\ 1 \leq p < \infty, \\ \sup\{|x(i)| : i \in \mathbb{N}\} & \text{if} \quad x \in \ell^\infty,\ p = \infty. \end{cases}$$

We have already seen that $\|\cdot\|_\infty$ is a norm on ℓ^∞. It can be easily seen that ℓ^1 is a linear space and $\|\cdot\|_1$ is a norm on ℓ^1.

Next, we show that ℓ^p is a linear space and $\|\cdot\|_p$ is a norm on ℓ^p for $1 < p < \infty$. For this, let $x, y \in \ell^p$. Then, for any $n \in \mathbb{N}$, we know by Theorem 1.2.5 that

$$\begin{aligned}\Big(\sum_{i=1}^{n} |x(i)+y(i)|^p\Big)^{1/p} &\leq \Big(\sum_{i=1}^{n} |x(i)|^p\Big)^{1/p} + \Big(\sum_{i=1}^{n} |y(i)|^p\Big)^{1/p} \\ &\leq \|x\|_p + \|y\|_p\end{aligned}$$

so that

$$\sum_{i=1}^{n} |x(i)+y(i)|^p \leq (\|x\|_p + \|y\|_p)^p.$$

Hence, taking the limit, we obtain

$$\sum_{i=1}^{\infty} |x(i)+y(i)|^p \leq (\|x\|_p + \|y\|_p)^p.$$

Thus, for every $x, y \in \ell^p$ and $\alpha \in \mathbb{K}$,

$$x + y \in \ell^p \quad \text{and} \quad \alpha x \in \ell^p$$

so that ℓ^p is a linear space and $x \mapsto \|x\|_p$ satisfies the triangle inequality. All other conditions for $\|\cdot\|_p$ to be a norm can be easily verified. $\square$

Remark 1.2.3 We observe the following:

1. For $1 \leq p < r \leq \infty$, ℓ^p is a proper subspace of ℓ^r.
2. As in the case of $\mathbb{K}^n$ for $n \geq 2$ (cf. Theorem 1.2.5), we see that for $0 < p < 1$,
$$\|x\|_p := \Big(\sum_{i=1}^{\infty} |x(i)|^p\Big)^{1/p}$$
does not define a norm on $\{x \in \mathcal{F}(\mathbb{N}) : \sum_{i=1}^{\infty} |x(i)|^p < \infty\}$.

$\Diamond$

Example 1.2.7 Recall the following subspaces of $\mathcal{F}(\mathbb{N})$:

$$\begin{aligned} c_{00} &= \{x \in \mathcal{F}(\mathbb{N}) : \exists\, k \in \mathbb{N} \text{ such that } x(n) = 0\,\forall\, n \geq k\}, \\ c_0 &= \{x \in \mathcal{F}(\mathbb{N}) : x(n) \to 0 \text{ as } n \to \infty\}, \\ c &= \{x \in \mathcal{F}(\mathbb{N}) : (x(n)) \text{ converges }\}. \end{aligned}$$

We have the strict inclusions

$$c_{00} \subset \ell^p \subset c_0 \subset c \subset \ell^\infty$$

for $1 \le p < \infty$. Further, for $1 \le p \le \infty$, $\|\cdot\|_p$ is a norm on c_{00}, and $\|\cdot\|_\infty$ is a norm on c_0 and c. □

The following lemma helps us to have norms on an arbitrary linear space using some other normed linear spaces.

Lemma 1.2.7 *Let X be a linear space, Y be a normed linear space with norm $\|\cdot\|_Y$, and $T : X \to Y$ be an injective linear operator. Then*

$$\|x\|_X := \|T(x)\|_Y, \qquad x \in X,$$

defines a norm on X.

Proof. Clearly, $\|x\|_X \ge 0$ and since T is injective $\|x\|_X = 0$ if and only if $x = 0$. Also by the linearity of T, for every $x, y \in X$,

$$\begin{aligned} \|x+y\|_X &= \|T(x+y)\|_Y = \|T(x)+T(y)\|_Y \\ &\le \|T(x)\|_Y + \|T(y)\|_Y = \|x\|_X + \|y\|_X, \end{aligned}$$

and for $x \in X$, $\alpha \in \mathbb{K}$,

$$\|\alpha x\|_X = \|T(\alpha x)\|_Y = \|aT(x)\|_Y = |\alpha|\,\|T(x)\|_Y = |\alpha|\,\|x\|_X.$$

Thus, $x \mapsto \|x\|_X$ is a norm on X. ∎

As a particular case of Lemma 1.2.7, we have the following example.

Example 1.2.8 Let X be a finite dimensional linear space and $E = \{u_1, \ldots, u_n\}$ be an ordered basis of X. Let $\|\cdot\|$ be norm on $\mathbb{K}^n$, e.g., $\|\cdot\| = \|\cdot\|_p$ for $1 \le p \le \infty$. Let $T : X \to \mathbb{K}^n$ be defined by

$$T(x) = (\alpha_1, \ldots, \alpha_n), \qquad x \in X,$$

where $(\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$ is such that $x = \sum_{i=1}^n \alpha_i u_i$. Then T is an injective linear operator. Hence, by Lemma 1.2.7,

$$\|x\|_E := \|Tx\|, \qquad x \in X,$$

defines a norm on X. □

Next subsection gives a special class of normed linear spaces whose norms are induced by a new structure on the linear spaces, the so called *inner products*.

1.2.3 Inner product spaces

Definition 1.2.3 An **inner product** on a linear space X is a map which associates each pair (x, y) of vectors from X, a unique scalar denoted by $\langle x, y\rangle$ such that the following conditions are satisfied:

(i) For every $x \in X$, $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for every $x, y \in X$,

(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in X$,

(iv) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for every $x, y \in X$ and $\alpha \in \mathbb{K}$.

A linear space X together with an inner product is called an **inner product space**. $\Diamond$

We show that if X is an inner product space, then the inner product will induce a norm on X. For showing this, we require the analogue of Schwarz inequality proved for the spaces $\mathbb{K}^n$ and $C[a, b]$.

Theorem 1.2.8 (Cauchy-Schwarz inequality) *Let X be an inner product space. If $x, y \in X$, then*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \qquad \forall\, x, y \in X.$$

Proof. Let $x, y \in X$. Clearly, the inequality holds if one of x and y is 0. Hence, assume that both x and y are non-zero. Thus, by taking

$$u = \frac{x}{\sqrt{\langle x, x\rangle}}, \qquad v = \frac{y}{\sqrt{\langle y, y\rangle}},$$

we have $\langle u, u\rangle = 1 = \langle v, v\rangle$. Hence,

$$\begin{aligned} 0 &\leq \big\langle u - \langle u, v\rangle v,\ u - \langle u, v\rangle v\big\rangle \\ &\leq \langle u, u\rangle - \overline{\langle u, v\rangle}\langle u, v\rangle - \langle u, v\rangle\overline{\langle u, v\rangle} + \langle u, v\rangle\overline{\langle u, v\rangle}|\langle v, v\rangle| \\ &= 1 - |\langle u, v\rangle|^2. \end{aligned}$$

Thus, $|\langle u, v\rangle|^2 \leq 1$. But,

$$\langle u, v\rangle = \frac{\langle x, y\rangle}{\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}}.$$

Thus, we have proved the required inequality. ∎

Theorem 1.2.9 *Let X be an inner product space and for $x \in X$, let*

$$\|x\| := \sqrt{\langle x, x\rangle}, \qquad \forall\, x \in X.$$

Then $\|\cdot\|$ is a norm on X.

Proof. Conditions for $\|\cdot\|$ to be a norm, except the triangle inequality, can be verified easily. To show the triangle inequality, let $x, y \in X$. Then,

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle.$$

Now, by Schwarz inequality,

$$|\mathrm{Re}\langle x, y\rangle| \le |\langle x, y\rangle| \le \|x\|\,\|y\|.$$

Hence,

$$\|x+y\|^2 \le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| = (\|x\| + \|y\|)^2.$$

Thus,we obtain, $\|x+y\| \le \|x\| + \|y\|$. ∎

Definition 1.2.4 An inner product space which is a complete with respect to the norm induced by the inner product is called a **Hilbert space**. ♢

Example 1.2.9 As examples of inner product spaces, we have the following:

(i) $X = \mathbb{K}^n$ with $\langle x, y\rangle = \displaystyle\sum_{i=1}^{n} x(i)\overline{y(i)}$.

(ii) $X = \ell^2$ with $\langle x, y\rangle = \displaystyle\sum_{i=1}^{\infty} x(i)\overline{y(i)}$.

(iii) $X = C[a, b]$ with $\langle x, y\rangle = \displaystyle\int_a^b x(t)\overline{y(t)}\,dt$.

Note that, in the above three examples, the norm induced by the inner product is $\|\cdot\|_2$. We shall see that with respect to the given inner products, $\mathbb{K}^n$ and ℓ^2 are Hilbert spaces, whereas $C[a, b]$ is not a Hilbert space. □

1.3 Banach Spaces

Recall that a normed linear space is said to be a *Banach space* if the metric induced by the norm is complete. Banach spaces play important and useful roles in analysis and its applications. In this section we shall consider some standard examples of Banach spaces and discuss some properties of Banach spaces.

1.3.1 Examples and properties

First and foremost we recall that $\mathbb{K}$ is a Banach space with respect to the usual norm, namely, the absolute value.

Example 1.3.1 We show that for $k \in \mathbb{N}$, $\mathbb{K}^k$ with $\|\cdot\|_p$ is a Banach space for any p with $1 \le p \le \infty$. Although the reader might be knowing that

$$\|x\|_\infty = \max\{|x(i)| : i = 1, \ldots, k\}$$

defines a complete norm on $\mathbb{K}^k$, we provide its proof for the sake of completion of presentation. For this, let (x_n) be a Cauchy sequence with respect to $\|\cdot\|_\infty$, i.e., for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_\infty < \varepsilon \quad \forall n, m \ge N.$$

Hence, for each $i \in \{1, \ldots, k\}$, $(x_n(i))$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, there exists $\alpha_i \in \mathbb{K}$ such that $x_n(i) \to \alpha_i$ as $n \to \infty$. Hence, with $x = (\alpha_1, \ldots, \alpha_k)$,

$$\|x_n - x\| = \max\{|x_n(i) - \alpha_i| : i = 1, \ldots, k\} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\mathbb{K}^k$ is a Banach space with respect to the norm $\|\cdot\|_\infty$.

Next we observe that for any p with $1 \le p < \infty$ and $x \in \mathbb{K}^k$,

$$\|x\|_\infty \le \|x\|_p \le k^{1/p}\|x\|_\infty.$$

hence, for every sequence (x_n) in $\mathbb{K}^k$,

- (x_n) is a Cauchy sequence with respect to $\|\cdot\|_p$ if and only if (x_n) is a Cauchy sequence with respect to $\|\cdot\|_\infty$,
- (x_n) converges to $x \in \mathbb{K}^k$ with respect to $\|\cdot\|_p$ if and only if (x_n) converges to x with respect to $\|\cdot\|_\infty$.

Hence, $\mathbb{K}^k$ is a Banach space with respect to $\|\cdot\|_p$ for any p with $1 \le p \le \infty$. □

Before proceeding to discuss further, let us give some examples of normed linear spaces which are not Banach spaces.

Example 1.3.2 Let $X_p := c_{00}$ with $\|\cdot\|_p$. For $1 \le p \le \infty$, let

$$x_n(j) = \begin{cases} \frac{1}{j^2}, & 1 \le j \le n, \\ 0, & j > n. \end{cases}$$

Note that $x_n \in X_p$ for all $n \in \mathbb{N}$. It can be seen that (x_n) is a Cauchy sequence in X_p which does not converge to any element in X_p. Thus, X_p is not a Banach space.

In fact, c_{00} is a dense proper subspace of ℓ^p for $1 \le p < \infty$, and it is a dense subspace of c_0 with respect to $\|\cdot\|_\infty$. Indeed, for $x \in \mathcal{F}(\mathbb{N})$, considering the sequence (x_n) defined by

$$x_n(j) = \begin{cases} x(j), & j \le n, \\ 0, & j > n, \end{cases}$$

we see that $x_n \in c_{00}$ for all $n \in \mathbb{N}$, and $\|x - x_n\|_p \to 0$ if $x \in \ell^p$ for $1 \le p < \infty$, and if $x \in c_0$ for $p = \infty$. □

Example 1.3.3 Let $X = C[a, b]$ be with $\|\cdot\|_1$. Let $c \in (a, b)$ and $k \in \mathbb{N}$ be such that $c + \frac{1}{n} < b$ for all $n \ge k$. For $n \ge k$, let

$$x_n(t) = \begin{cases} 1, & a \le t \le c \\ 1 - n(t - c), & c < t \le c + \frac{1}{n}, \\ 0, & c + \frac{1}{n} \le t \le b. \end{cases}$$

It can be seen that $x_n \in X$ for all $n \ge k$, (x_{k+n}) is a Cauchy sequence in X which does not converge to any element in X. Thus, X is not a Banach space. □

In Example 1.3.1, for proving that $\mathbb{K}^k$ is a Banach space with respect to $\|\cdot\|_p$ for $1 \le p < \infty$, what we used is the fact that $\mathbb{K}^k$ is a Banach space with respect to $\|\cdot\|_\infty$ and the inequality

$$\|x\|_\infty \le \|x\|_p \le k^{1/p}\|x\|_\infty \quad \forall\, x \in \mathbb{K}^k.$$

Definition 1.3.1 Let X be a linear space with norms $\|\cdot\|$ and $\|\cdot\|_*$.

1. The norms $\|\cdot\|$ and $\|\cdot\|_*$ are said to be **equivalent** if there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x\| \le \|x\|_* \le c_2\|x\| \quad \forall\, x \in X.$$

2. The norm $\|\cdot\|$ is said to be **weaker** than $\|\cdot\|_*$ if there exists $c > 0$ such that

$$\|x\| \le c\|x\|_* \quad \forall\, x \in X,$$

and in that case $\|\cdot\|_*$ is said to be **stronger** than $\|\cdot\|$.

$\Diamond$

Theorem 1.3.1 *Suppose $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent norms on a linear space X. Then X is a Banach space with respect to $\|\cdot\|$ if and only if X is a Banach space with respect to $\|\cdot\|_*$.*

Proof. Exercise. ∎

In fact, we have the following more general result.

Theorem 1.3.2 *Let X and Y be a normed linear spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively and let $T : X \to Y$ be a bijective linear operator such that there exist positive constants c_1, c_2 satisfying*

$$c_1\|x\|_X \le \|Tx\|_Y \le c_2\|x\|_X \qquad \forall\, x \in X. \tag{$*$}$$

Let $S \subseteq X$. Then the following hold:

(i) *X is complete if and only if Y is complete.*

(ii) *S is compact in X if and only if $T(S)$ is compact in Y.*

(iii) *S is closed in X if and only if $T(S)$ is closed in Y.*

(iv) *S is bounded in X if and only if $T(S)$ is bounded in Y.*

Proof. Recall that a subset Ω_0 of a metric space Ω is compact if and only every sequence in Ω_0 has a subsequence which converges to an element in Ω_0.

Now, let (x_n) and (y_n) be sequences in X and Y, respectively, such that $y_n = Tx_n$, and $x \in X$ and $y \in Y$ such that $y = Tx$. Then from the relation $(*)$, we observe that

1. (x_n) is a Cauchy sequence in X if and only if (y_n) is a Cauchy sequence in Y,

2. (x_n) converges to $x \in X$ if and only if (y_n) converges to $y \in Y$,

3. (x_n) has a convergent subsequence if and only if (y_n) has a convergent subsequence.

From these, (i), (ii) and (iii) follow, and (iv) is obvious. ▮

We have already noted that, for $p, r \in [1, \infty]$, the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ on $\mathbb{K}^k$ are equivalent. We shall show that *any two norms on a finite dimensional vector space are equivalent.* However, this need not be true for infinite dimensional case as the following example shows.

Example 1.3.4 Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $C[a, b]$, that is, for $x \in C[a, b]$,

$$\|x\|_1 := \int_0^1 |x(t)|dt, \qquad \|x\|_\infty = \sup_{0 \le t \le 1} |x(t)|.$$

We see immediately that

$$\|x\|_1 \le (b-a)\|x\|_\infty \qquad \forall\, x \in C[a, b].$$

However, there is no $c > 0$ such that

$$\|x\|_\infty \le c\|x\|_1 \qquad \forall\, x \in C[a, b].$$

To see this we consider the sequence (x_n) in $C[a, b]$ such that for $n > 2/(b-a)$,

$$x_n(t) = \begin{cases} n, & a \le t \le a + \frac{2}{n}, \\ 0, & a + \frac{2}{n} < t \le b. \end{cases}$$

Then we have

$$\|x_n\|_\infty = n \quad \text{and} \quad \|x_n\|_1 = 1 \qquad \forall\, n > 2/(b-1).$$

□

Theorem 1.3.3 *The following hold.*

(i) *Any two norms on a finite dimensional space are equivalent.*

(ii) *Every finite dimensional normed linear space is a Banach space.*

(iii) *Every finite dimensional subspace of a normed linear space is closed.*

Proof. Let X be a normed linear space of dimension n with $\|\cdot\|$ and $E := \{u_1, \ldots, u_n\}$ be an ordered basis of X. Let $T : X \to \mathbb{K}^n$ be defined by

$$Tx = (\alpha_1, \ldots, \alpha_n), \qquad x = \sum_{i=1}^{n} \alpha_i u_i \in X.$$

Let

$$\|x\|_E := \|Tx\|_\infty, \qquad x \in X.$$

Then we know that $\|\cdot\|_E$ is a norm on X and since $\mathbb{K}^n$ is complete with respect to the metric induced by $\|\cdot\|_\infty$, X is a Banach space with respect to $\|\cdot\|_E$ as well (cf. Lemma 1.2.7 and Theorem 1.3.2). Also we know from real analysis that every complete subset of a metric space is closed. Hence, by Theorem 1.3.2, it is enough to show that $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent.

Note that

$$\|x\| \le \sum_{i=1}^{n} |\alpha_i(x)|\, \|u_i\| \le \|x\|_E \sum_{i=1}^{n} \|u_i\|. \tag{$*$}$$

Thus, $\|\cdot\|$ is weaker than $\|\cdot\|_E$.

Now, let $S = \{x \in X : \|x\|_E = 1\}$. Then we see that

$$T(S) = \{y \in \mathbb{K}^n : \|y\|_\infty = 1\}.$$

Since $T(S)$ is a compact subset of $\mathbb{K}^n$ (with respect tot the norm $\|\cdot\|_\infty$), by Theorem 1.3.2, $S = \{x \in X : \|x\|_E = 1\}$ is compact in X. Also, note that, by $(*)$, the function $x \mapsto \|x\|$ is continuous on X with respect to $\|\cdot\|_E$. Hence, there exists $x_0 \in S$ such that

$$\|x_0\| = \inf\{\|x\| : \|x\|_E = 1\}.$$

Now, let $x \in X$ be a non-zero vector in X. Consider the vector $u = x/\|x\|_E$. Then $\|u\|_E = 1$ and hence $\|x_0\| \le \|u\|$, that is,

$$\|x\|_E \|x_0\| \le \|x\| \qquad \forall\, x \in X.$$

This, together with $(*)$, shows that $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent. ∎

Remark 1.3.1 The proof of equivalence of norms in the proof of Theorem 1.3.4 is adapted from [1]. ◊

Theorem 1.3.4 (Heine-Borel theorem) *Every closed and bounded subset of a finite dimensional normed linear space is compact.*

Proof. Suppose X is finite dimensional, say $\dim(X) = m$. Let $E := \{u_1, \ldots, u_m\}$ be an ordered basis of X. Define $T : X \to \mathbb{K}^m$ by

$$Tx = (\alpha_1, \ldots, \alpha_m), \quad x = \sum_{j=1}^{m} \alpha_j u_j \in X,$$

and let

$$\|x\|_E := \|Tx\|_\infty, \quad x \in X.$$

We know that $\|\cdot\|_E$ is a norm on X. By Theorem 1.3.3, this norm is equivalent to the original norm $\|\cdot\|$ on X, that is, there exists $c_1, c_2 > 0$ such that

$$c_1\|x\| \le \|Tx\|_\infty \le c_2\|x\| \quad \forall\, x \in X.$$

Now, let S be a closed and bounded subset of X. By Theorem 1.3.2, $T(S)$ is closed and bounded in $\mathbb{K}^m$, and hence $T(S)$ is compact in $\mathbb{K}^m$. Again, by Theorem 1.3.2, S is compact in X. ∎

From Theorem 1.3.4 we deduce the following best approximation property for finite dimensional subspaces.

Theorem 1.3.5 (A best approximation theorem) *Let X be a normed linear space and X_0 be a finite dimensional subspace of X. Then for every $x \in X$, there exists $x_0 \in X_0$ such that*

$$\|x - x_0\| = \operatorname{dist}(x, X_0).$$

Further, if X_0 is a proper subspace, then there exists $\tilde{x} \in X$ such that

$$\|\tilde{x}\| = 1 \quad \textit{and} \quad \operatorname{dist}(\tilde{x}, X_0) = 1.$$

Proof. Let $x \in X$ and $d := \operatorname{dist}(x, X_0)$. Then there exists a sequence (x_n) in X_0 such that

$$\|x - x_n\| \to d \quad \text{as} \quad n \to \infty.$$

In particular, (x_n) is a bounded sequence in X_0. Since X_0 is finite dimensional, by Theorem 1.3.4, (x_n) has a subsequence, say $(\tilde{x}_n)$, which converges to some element $x_0 \in X_0$. Hence, we have

$$\|x - \tilde{x}_n\| \to \|x - x_0\| \quad \text{as} \quad n \to \infty,$$

so that $\|x - x_0\| = \text{dist}\,(x, X_0)$.

Next suppose that $X_0 \neq X$ and $x \in X \setminus X_0$. Then

$$\tilde{x} := \frac{x - x_0}{\|x - x_0\|}$$

satisfies $\|\tilde{x}\| = 1$ and $\text{dist}\,(\tilde{x}, X_0) = 1$. ∎

By Theorem 1.3.4, in a finite dimensional normed linear space, every closed and bounded subset is compact. In fact, it is a characterization of the finite dimensionality of a normed linear space, as Theorem 1.3.7 shows.

Theorem 1.3.6 *Let X be a normed linear space. Then X is finite dimensional if and only if the set $S := \{x \in X : \|x\| = 1\}$ is totally bounded.*

Proof. Suppose X is finite dimensional. Then, by Theorem 1.3.4, $S := \{x \in X : \|x\| = 1\}$ is compact. In particular S is totally bounded.

To prove the converse, assume that X is infinite dimensional. We have to show that S is not totally bounded. For this, let x_0 be a nonzero element in X, and $X_0 := \text{span}\,\{x_0\}$. Since X_0 is a finite dimensional subspace of X, by Theorem 1.3.5, there exists $x_1 \in X$ such that

$$\|x_1\| = 1 \quad \text{and} \quad \text{dist}\,(x_1, X_0) = 1.$$

Now, let $X_1 := \text{span}\,\{x_0, x_1\}$. Again, since X_1 is a finite dimensional subspace of X, Theorem 1.3.5 implies the existence of $x_2 \in X$ such that

$$\|x_2\| = 1 \quad \text{and} \quad \text{dist}\,(x_2, X_1) = 1.$$

Now, by induction argument, we obtain a sequence (x_n) in X such that

$$\|x_n\| = 1 \quad \text{and} \quad \text{dist}\,(x_n, X_{n-1}) = 1 \quad \forall n \in \mathbb{N},$$

where $X_n := \text{span}\,\{x_0, x_1, \ldots, x_n\}$ for each $n \in \mathbb{N}$. Since $x_k \in X_{n-1}$ for $n > k$, we have

$$\|x_n - x_m\| \geq 1 \quad \forall\, n, m \in \mathbb{N};\; n \neq m.$$

In particular, the sequence (x_n) in $S := \{x \in X : \|x\| = 1\}$ cannot have a Cauchy subsequence. Hence, S is not totally bounded. This completes the proof. ∎

Using Theorem 1.3.4 and Theorem 1.3.6, we derive the following theorem.

Theorem 1.3.7 *Let X be a normed linear space. Then the following are equivalent.*

(i) *X is finite dimensional.*

(ii) *Every closed and bounded subset of X is compact.*

(iii) *The closed unit ball $D := \{x \in X : \|x\| \leq 1\}$ is compact.*

(iv) *For any $x_0 \in X$ and $r > 0$, the open ball $B(x_0, r)$ is totally bounded.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) follow from Theorem 1.3.5 and using the facts that every closed subset of a compact set is compact and every subset of a compact set is totally bounded. Therefore, it is enough to show the implication (iv) $\Longrightarrow$ (i). So, let us assume that $B(x_0, r)$ is totally bounded for some $x_0 \in X$ and $r > 0$. Since

$$x_0 + \frac{r}{2}\{x \in X : \|x\| = 1\} \subseteq B(x_0, r),$$

it follows that $\{x \in X : \|x\| = 1\}$ is also totally bounded. Hence, by Theorem 1.3.6, X is finite dimensional. ∎

Now, we give some examples of infinite dimensional Banach spaces.

Example 1.3.5 Let Ω be a nonempty set. We show that $B(\Omega)$ is a Banach space. For this, let (x_n) be a Cauchy sequence in $B(\Omega)$ and let $\varepsilon > 0$ be given. We have to show that there exists $x \in B(\Omega)$ such that $\|x_n - x\|_\infty \to 0$ as $n \to \infty$. Now, since (x_n) is a Cauchy sequence in $B(\Omega)$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_\infty < \varepsilon \quad \forall\, n, m \geq N.$$

In particular, for each $t \in \Omega$,

$$|x_n(t) - x_m(t)| \leq \|x_n - x_m\|_\infty < \varepsilon \quad \forall\, n, m \geq N. \qquad (*)$$

Hence, for each $t \in \Omega$, $(x_n(t))$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, there exists $\alpha_t \in \mathbb{K}$ such that

$$x_n(t) \to \alpha_t \qquad \text{as} \quad n \to \infty.$$

Define $x(t) := \alpha_t$, $t \in \Omega$. Then, by $(*)$, we have

$$|x_n(t) - x(t)| = \lim_{m\to\infty} |x_n(t) - x_m(t)| \le \varepsilon \quad \forall\, n \ge N.$$

This is true for all $t \in \Omega$. Hence, it follows that

$$x \in B(\Omega) \quad \text{and} \quad \|x_n - x\|_\infty \le \varepsilon \quad \forall\, n \ge N.$$

Thus, (x_n) converges to $x \in B(\Omega)$. □

Example 1.3.6 Let Ω be a metric space. We show that $C_b(\Omega)$, the set of all $\mathbb{K}$-valued bounded continuous functions defined on Ω, is a closed subspace of $B(\Omega)$ so that it is a Banach space with respect to the norm $\|\cdot\|_\infty$. For this, let (x_n) be a sequence in $C_b(\Omega)$ such that $\|x_n - x\|_\infty \to 0$ for some $x \in B(\Omega)$. We have to show that $x \in C_b(\Omega)$. That is, to show that x is continuous at every $\tau \in \Omega$. So, let $\tau \in \Omega$ and $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that

$$\|x_n - x\|_\infty < \varepsilon \quad \forall\, n \ge N.$$

Hence, for $t \in \Omega$ and for $n \ge N$,

$$\begin{aligned} |x(t) - x(\tau)| &\le |x(t) - x_n(t)| + |x_n(t) - x_n(\tau)| + |x_n(\tau) - x(\tau)| \\ &\le \|x - x_n\|_\infty + |x_n(t) - x_n(\tau)| + \|x_n - x\|_\infty \\ &\le \varepsilon + |x_n(t) - x_n(\tau)| + \varepsilon. \end{aligned}$$

In particular,

$$|x(t) - x(\tau)| \le \varepsilon + |x_N(t) - x_N(\tau)| + \varepsilon.$$

Since $X_N \in C_b(\Omega)$, there exists an open set $G \subseteq \Omega$ containing τ such that

$$|x_N(t) - x_N(\tau)| < \varepsilon \quad \forall\, t \in G.$$

Thus, we have proved that for every $\varepsilon > 0$, there exists an open set G containing τ such that

$$|x(t) - x(\tau)| < 3\varepsilon \quad \forall\, t \in G.$$

Therefore, $x \in C_b(\Omega)$. □

Example 1.3.7 We have seen in Example 1.3.5 that for every nonempty set Ω, $B(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_\infty$. In particular, $\ell^\infty := \ell^\infty(\mathbb{N})$ is a Banach space. Now, we show that ℓ^p is a Banach space for $1 \le p < \infty$ as well. For this, let $1 \le p < \infty$ and let (x_n) be a Cauchy sequence in ℓ^p and let $\varepsilon > 0$ be given. We have to show that there exists $x \in \ell^p$ such that $\|x_n - x\|_p \to 0$ as $n \to \infty$. Since (x_n) is a Cauchy sequence in ℓ^p, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_p < \varepsilon \quad \forall\, n, m \ge N.$$

In particular, for each $k \in \mathbb{N}$,

$$\sum_{j=1}^{k} |x_n(j) - x_m(j)|^p \le \|x_n - x_m\|_p^p < \varepsilon^p \quad \forall\, n, m \ge N. \qquad (*)$$

Hence, for each $j \in \mathbb{N}$, $(x_n(j))$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, there exists $\alpha_j \in \mathbb{K}$ such that

$$x_n(j) \to \alpha_j \qquad \text{as} \quad n \to \infty.$$

Define $x(j) := \alpha_j$, $j \in \Omega$. Then, by $(*)$, for every $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{k} |x_n(j) - x(j)|^p = \lim_{m\to\infty} \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p \le \varepsilon^p \quad \forall\, n \ge N.$$

This is true for every $k \in \mathbb{N}$. Hence,

$$\sum_{j=1}^{\infty} |x_n(j) - x(j)|^p \le \varepsilon^p \quad \forall\, n \ge N.$$

Therefore, it follows that

$$x \in \ell^p \quad \text{and} \quad \|x_n - x\|_p < \varepsilon \quad \forall\, n \ge N.$$

Thus, (x_n) converges to $x \in \ell^p$. $\square$

Example 1.3.8 Following the arguments as in Example 1.3.6, it can be shown that

(i) c, the space of all convergent scalar sequences, is a closed subspace of ℓ^∞.

(ii) c_0, the space of all convergent scalar sequences having limit 0, is a closed subspace of c.

Thus, both c and c_0 with $\|\cdot\|_\infty$ are Banach spaces. $\square$

We have seen that c_{00} is not a Banach space with respect to $\|\cdot\|_p$ for any p with $1 \leq p \leq \infty$. One may ask whether c_{00} is a Banach space with respect to any other norm. The answer is, in fact, negative, which we shall see in the next subsection. In fact, we shall deduce this fact from a more general result. For its proof, we shall we shall make use of the *Baire category theorem* (cf. [9]), which the reader must have already seen in the real analysis.

Theorem 1.3.8 (Baire category theorem) *A complete metric space cannot be written as a countable union of closed subsets with empty interiors.*

We shall also make use of the following lemma.

Lemma 1.3.9 *If a subspace of a normed linear space contains any of its interior point, then it is the whole space.*

Proof. Let X_0 be a subspace of a normed linear space X and x_0 be an interior point of X. Then there is $r > 0$ such that $B(x_0, r) \subset X_0$. Hence, for any non-zero $x \in X$,

$$x_0 + \frac{rx}{2\|x\|} \in B(x_0, r) \subset X_0.$$

Hence, it follows that $x \in X_0$, which implies that $X_0 = X$. ∎

Theorem 1.3.10 *A normed linear space with a denumerable basis cannot be a Banach space.*

Proof. Suppose X is a normed linear space having a denumerable basis, say $E = \{u_1, u_2, \ldots\}$. For $n \in \mathbb{N}$, let

$$X_n := \text{span}\,\{u_1, \ldots, u_n\}.$$

Then each X_n is a proper closed subspace of the space X_{n+1}, and

$$X = \bigcup_{n=1}^{\infty} X_n.$$

By Lemma 1.3.9, interior of each X_n is non-empty. Hence, by Baire-Category theorem (Theorem 1.3.8), X cannot be a Banach space. ∎

Following are some of the consequences of the above theorem:

1. The space c_{00} is not a Banach space with respect to any norm.
2. The space $\mathcal{P}$ of all polynomials is not a Banach space with respect to any norm.
3. Every infinite dimensional space has a subspace which is not a Banach space with respect to any norm.
4. A Banach space is finite dimensional if and only if every subspace of it is closed.

As in the case of series of scalars, we can define a series of vectors.

Definition 1.3.2 Let (x_n) be a sequence in a normed linear space X.

1. A **series** associated with (x_n) is an expression of the form $\sum_{n=1}^{\infty} x_n$, and for each $n \in \mathbb{N}$, the $s_n := \sum_{i=1}^{n} x_i$ is called the n^{th} **partial sum** of the series $\sum_{n=1}^{\infty} x_n$.
2. The series $\sum_{n=1}^{\infty} x_n$ is said to be a **convergent series** in X if the sequence (s_n) of its partial sums converges, and in that case we write $\displaystyle\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$.
3. The series $\sum_{n=1}^{\infty} x_n$ is said to be an **absolutely convergent series** if $\sum_{n=1}^{\infty} \|x_n\|$ converges.

$\Diamond$

Recall that if a series of scalars is absolutely convergent, then it converges. However, this need not be true for series of vectors, as the following example shows.

Example 1.3.9 Let $X = c_{00}$ with any of the norms $\|\cdot\|_p$, $1 \le p \le \infty$. Let

$$x_n = \frac{e_n}{n^2}, \quad n \in \mathbb{N}.$$

Clearly, $\sum_{n=1}^{\infty} x_n$ is absolutely convergent as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. But, $\sum_{n=1}^{\infty} x_n$ does not converge in X. To see this, let

$$s_n = \sum_{j=1}^{n} \frac{e_j}{j^2}, \quad n \in \mathbb{N}.$$

Suppose there exists $x \in c_{00}$ such that $\|s_n - x\|_p \to 0$ as $n \to \infty$. Then we would get $\|s_n(j) - x(j)| \to 0$ as $n \to \infty$ for each $j \in \mathbb{N}$. Observe that

$$s_n(j) = 1/j^2 \quad \forall\, n \geq j.$$

Thus, we arrive at a contradiction that $x(j) = 1/j^2$ for all $j \in \mathbb{N}$. □

The situation in the above example does not occur if the space is a Banach space. That is the essence of the following theorem.

Theorem 1.3.11 *A normed linear space X is a Banach space if and only if every absolutely convergent series in X is convergent in X.*

Proof. Suppose X is a Banach space and (x_n) is a sequence of vectors in X such that $\sum_{i=1}^{\infty} \|x_i\|$ converges. Let $\varepsilon > 0$ be given. Then taking

$$s_n = \sum_{i=1}^{n} x_i, \quad n \in \mathbb{N},$$

we have for $n > m$,

$$\|s_n - s_m\| = \|\sum_{i=m+1}^{n} x_i\| \leq \sum_{i=m+1}^{n} \|x_i\|.$$

Since $\sum_{i=1}^{\infty} \|x_i\|$ converges, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=m}^{n} \|x_i\| < \varepsilon \qquad \forall\, n > m \geq N.$$

Thus, (s_n) is a Cauchy sequence in X. By the completeness of X, (s_n) converges, that is, the series $\sum_{i=1}^{\infty} x_i$ converges.

Conversely, suppose X is a normed linear space such that every absolutely convergent series in X is convergent in X. We show that X is Banach space. For this, let (y_n) be a Cauchy sequence in X. For this it is enough to show that there(y_n) has a convergent subsequence. For each $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be such that

$$\|y_n - y_m\| < \frac{1}{2^k} \qquad \forall n, m \geq n_k.$$

Without loss of generality, assume that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Then we have

$$\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k} \qquad \forall\, k \in \mathbb{N}.$$

In particular, the series $\sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\|$ converges. By the hypothesis, the series $\sum_{k=1}^{\infty}(y_{n_{k+1}} - y_{n_k})$ converges. That is, taking $z_m = \sum_{k=1}^{m}(y_{n_{k+1}} - y_{n_k})$, $m \in \mathbb{N}$, the sequence (z_m) converges. But,

$$y_{n_m} = y_{n_1} + z_{m-1} \qquad \forall, m \geq 2.$$

Hence, (y_{n_m}) converges. This completes the proof. ▮

1.3.2 Semi-norms and quotient spaces

In the definition of a norm if the condition

$$\|x\| = 0 \quad \Longrightarrow \quad x = 0$$

is not demanded, then the resulting function is called a *semi-norm*. More precisely, we have the following definition.

Definition 1.3.3 Let X be a linear space. A function $p : X \to \mathbb{R}$ is called a **semi-norm** if the following conditions are satisfied:

(i) $p(x + y) \leq p(x) + p(y) \quad \forall\, x, y \in X$,

(ii) $p(\alpha x) = |\alpha| p(x) \quad \forall\, x \in X,\ \alpha \in \mathbb{K}$.

◊

The following can be seen easily (*Exercise*):

- If p is a semi-norm on a linear space X, then $p(x) \geq 0$ for all $x \in X$ and $p(0) = 0$.
- Every norm on a linear space is a semi-norm.

Theorem 1.3.12 *Let X be a normed linear space and X_0 be a subspace of X. Then*

$$p(x) = \operatorname{dist}(x, X_0) := \inf\{\|x - u\| : u \in X_0\}$$

defines a seminorm on X.

Proof. For $x, y \in X$, and for $u, v \in X_0$, we have

$$\|x + y - (u + v)\| \leq \|x - u\| + \|y - v\|.$$

Hence,

$$\operatorname{dist}(x + y, X_0) \leq \|x - u\| + \|y - v\| \qquad \forall\, u, v \in X_0.$$

Now, taking infimum over all $u, v \in X_0$, we obtain

$$\text{dist}\,(x + y, X_0) \le \text{dist}\,(x, X_0) + \text{dist}\,(y, X_0).$$

We also have (verify!), for every $x \in X$, $\alpha \in \mathbb{K}$,

$$\text{dist}\,(\alpha x, X_0) = \inf\{\|\alpha x - \alpha u\| : u \in X_0\} = |\alpha| \text{dist}\,(x, X_0).$$

This completes the proof. ▮

Although a seminorm need not be a norm, a seminorm does give rise to a norm on a quotient space in a natural way as Theorem 1.3.13 below shows. First let us recall the definition of a quotient space.

Let X be a linear space and X_0 be a subspace of X. Consider the quotient space X/X_0, i.e.,

$$X/X_0 := \{x + X_0 : x \in X\},$$

where $x + X_0 := \{x + u : u \in X_0\}$ is the *coset* corresponding to x. Note that X/X_0 is the set of all equivalence classes of vectors in X under the equivalence relation:

$$x \sim y \iff x - y \in X_0.$$

Thus, for each $x \in X$, then equivalence class $[x]$ of x is the set $x + X_0$, i.e.,

$$[x] = x + X_0.$$

Defining addition and scalar multiplication as

$$[x] + [y] := [x + y], \qquad \alpha[x] := [\alpha x]$$

for $x, y \in X$ and $\alpha \in \mathbb{K}$, it can be easily verified that X/X_0 is a linear space with its zero as $[0] = X_0$. In order that the above operations to be meaningful, one has to verify that

$$u \in [x],\; v \in [y] \Longrightarrow [u] + [v] = [x + y].$$

Theorem 1.3.13 *Let $p : X \to \mathbb{R}$ be a seminorm on a linear space X. Then*

$$X_0 := \{x \in X : p(x) = 0\}$$

is a subspace of X, and the function

$$[x] \mapsto p(x)$$

is a norm on X/X_0.

Proof. The fact that $X_0 := \{x \in X : p(x) = 0\}$ is a subspace of X is obvious from the definition of a seminorm. Thus, it is enough to show that, for $x \in X$, $p(x) = 0$ implies $[x] = [0]$. Clearly, for $x \in X$,

$$p(x) = 0 \Longrightarrow x \in X_0 = [0] \Longrightarrow [x] = [0].$$

This completes the proof. ▮

Theorem 1.3.14 *Let X be a normed linear space and X_0 be a closed subspace of X. Then*

$$[x] \mapsto \text{dist}\,(x, X_0)$$

is a norm on X/X_0.

Proof. Let $p(x) = \text{dist}\,(x, X_0)$, $x \in X$. Since X_0 is a closed subspace of X, we have for $x \in X$,

$$p(x) = 0 \iff x \in X_0,$$

so that $X_0 = \{x \in X : p(x) = 0\}$. Now the result follows from Theorem 1.3.12 and Theorem 1.3.13 . ▮

Theorem 1.3.15 *If X is a Banach space, then X/X_0 is Banach space.*

Proof. Suppose X is a Banach space. Let (ξ_n) be a Cauchy sequence in X/X_0, and let $\xi_n = [x_n]$ for $n \in \mathbb{N}$. We have to show that (ξ_n) converges to an element in X/X_0. For each $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be such that

$$\text{dist}\,(x_n - x_m, X_0) < \frac{1}{2^k} \quad \forall\, n, m \geq j_j.$$

Without loss of generality, assume that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Then we have

$$\text{dist}\,(x_{n_{k+1}} - x_{n_k}, X_0) < \frac{1}{2^k} \quad \forall\, k \in \mathbb{N}.$$

Therefore, for each $k \in \mathbb{N}$, there exists $u_k \in X_0$ such that

$$\|x_{n_{k+1}} - x_{n_k} - u_k\| < \frac{1}{2^k} \quad \forall\, k \in \mathbb{N}.$$

Let $v_1 = 0$ and $v_{k+1} = \sum_{i=1}^{k} u_i$ for $i \in \mathbb{N}$. Then $v_k \in X_0$, $u_k = v_{k+1} - v_k$ for every $k \in \mathbb{N}$, and

$$\|(x_{n_{k+1}} - v_{k+1}) - (x_{n_k} - v_k)\| < \frac{1}{2^k} \quad \forall\, k \in \mathbb{N}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, it follows that the sequence $(x_{n_k} - v_k)_{k=1}^{\infty}$ is a Cauchy sequence in X. By the completeness of X, there exists $x \in X$ such that

$$\|x_{n_k} - v_k - x\| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence

$$\begin{aligned} \operatorname{dist}(x_{n_k} - x, X_0) &= \operatorname{dist}(x_{n_k} - v_j - x, X_0) \\ &\le \|x_{n_k} - v_k - x\| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Thus, we have shown that the Cauchy sequence (ξ_n) has a subsequence which converges to $[x] \in X/X_0$, so that (ξ_n) also converges to $[x]$. ∎

Using the concept of a seminorm, we prove the following.

Theorem 1.3.16 *If X is a normed linear space, then there exists a Banach space Y such that X is linearly isometric with a dense subspace of Y.*

Proof. Let X be a normed linear space with norm $\|\cdot\|$, and let $\mathcal{X}$ be the set of all Cauchy sequences in X. We observe that

$$(x_n), (y_n) \in \mathcal{X} \Longrightarrow (x_n + y_n) \in \mathcal{X},$$

$$(x_n) \in \mathcal{X},\ \alpha \in K \Longrightarrow (\alpha x_n) \in \mathcal{X}.$$

Now, for $(x_n), (y_n) \in \mathcal{X}$ and $\alpha \in K$, define

$$(x_n) + (y_n) := (x_n + y_n), \quad \alpha(x_n) := (\alpha x_n).$$

Then, it can seen that, with respect to these operations, $\mathcal{X}$ is a linear space with its zero as (θ_n) with $\theta_n = 0$ for all $n \in \mathbb{N}$, and $-(x_n) := (-x_n)$. Next we observe that

$$(x_n) \in \mathcal{X} \Longrightarrow \lim_{n\to\infty} \|x_n\| \text{ exists.}$$

This follows from the fact that if (x_n) is a Cauchy sequence, then $(\|x_n\|)$ is a Cauchy sequence in $\mathbb{R}$. Now, define $p : \mathcal{X} \to \mathbb{R}$ by

$$p(\tilde{x}) = \lim_{n\to\infty} \|x_n\|, \qquad \tilde{x} := (x_n) \in \mathcal{X}.$$

Then $p(\cdot)$ is a seminorm on $\mathcal{X}$. This follows from the relations

$$\lim_{n\to\infty} \|x_n + y_n\| \le \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\|,$$

$$\lim_{n\to\infty} \|\alpha x_n\| = |\alpha| \lim_{n\to\infty} \|x_n\|$$

for any $(x_n), (y_n) \in \mathcal{X}$ and $\alpha \in \mathbb{K}$. Hence, by Theorem 1.3.13,

$$\|[\tilde{x}]\|_* := p(\tilde{x}), \qquad [\tilde{x}] \in \mathcal{X}/\mathcal{X}_0,$$

defines a norm on the quotient linear space $\mathcal{X}/\mathcal{X}_0$, where

$$\mathcal{X}_0 := \{\tilde{x} \in \mathcal{X} : p(\tilde{x}) = 0\}.$$

Let $T : X \to \mathcal{X}/\mathcal{X}_0$ be defined by

$$T(x) = [\tilde{x}], \quad \text{where} \quad \tilde{x} := (x_n) \text{ with } x_n = x \quad \forall\, n \in \mathbb{N}.$$

We show that $\mathcal{X}/\mathcal{X}_0$ is a Banach space with respect to the norm $\|\cdot\|_*$ defined above, and T is a linear isometry with its range dense in $\mathcal{X}/\mathcal{X}_0$.

It can be easily seen that T is a linear operator, and

$$\|T(x)\|_* = p(\tilde{x}) = \|x\|.$$

Now, to show that range of T is dense in $\mathcal{X}/\mathcal{X}_0$, let $\tilde{x} = (x_n) \in \mathcal{X}$ and let $\varepsilon > 0$ be given. For each $k \in \mathbb{N}$, let

$$\tilde{x}_k := (x_{k,n}) \text{ with } x_{k,n} = x_k \quad \forall\, n \in \mathbb{N}.$$

Then

$$p(\tilde{x} - \tilde{x}_k) = \lim_{n\to\infty} \|x_n - x_{k,n}\| = \lim_{n\to\infty} \|x_n - x_k\|.$$

Let $N \in \mathbb{N}$ be such that $\|x_k - x_n\| < \varepsilon$ for all $k, n \ge N$. Hence,

$$\|[\tilde{x}] - T(x_k)\|_* = \|[\tilde{x}] - [\tilde{x}_k]\|_* = p(\tilde{x} - \tilde{x}_k) \le \varepsilon \qquad \forall k \ge N.$$

Thus, range of T is dense in $\mathcal{X}/\mathcal{X}_0$.

It remains to show that $\mathcal{X}/\mathcal{X}_0$ is a Banach space. For this, let $(\tilde{x}_n)$ be a sequence in $\mathcal{X}$ such that $([\tilde{x}_n])$ is a Cauchy sequence in $\mathcal{X}/\mathcal{X}_0$. Since range of T is dense in $\mathcal{X}/\mathcal{X}_0$, there exists (y_n) in X such that

$$\|[\tilde{x}_n] - [\tilde{y}_n]\|_* < \frac{1}{n} \qquad \forall\, n \in \mathbb{N},$$

where, for each $n \in \mathbb{N}$, $\tilde{y}_n = (y_{n,k})$ with $y_{n,k} = y_n$ for all $k \in \mathbb{N}$. Let $\tilde{y} = (y_n)$. We show that $\|[\tilde{x}_n] - [\tilde{y}]\|_* \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} \|[\tilde{x}_n] - [\tilde{y}]\|_* &\leq \|[\tilde{x}_n] - [\tilde{y}_n]\|_* + \|[\tilde{y}_n] - [\tilde{y}]\|_* \\ &< \frac{1}{n} + \|[\tilde{y}_n] - [\tilde{y}]\|_*, \end{aligned}$$

where

$$\|[\tilde{y}_n] - [\tilde{y}]\|_* = \lim_{k\to\infty} \|y_{n,k} - y_k\| = \lim_{k\to\infty} \|y_n - y_k\|.$$

But,

$$\begin{aligned} \|y_n - y_k\| &= \lim_{m\to\infty} \|y_{n,m} - y_{k,m}\| \\ &= \|[\tilde{y}_n] - [\tilde{y}_k]\|_* \\ &\leq \|[\tilde{y}_n] - [\tilde{x}_n]\|_* + \|[\tilde{x}_n] - [\tilde{x}_k]\|_* + \|[\tilde{x}_k] - [\tilde{y}_k]\|_* \\ &= \frac{1}{n} + \|[\tilde{x}_n] - [\tilde{x}_k]\|_* + \frac{1}{k}. \end{aligned}$$

Thus,

$$\|[\tilde{y}_n] - [\tilde{y}]\|_* = \lim_{k\to\infty} \|y_n - y_k\| \leq \frac{1}{n} + \lim_{k\to\infty} \|[\tilde{x}_n] - [\tilde{x}_k]\|_*$$

so that

$$\|[\tilde{x}_n] - [\tilde{y}]\|_* \leq \frac{1}{n} + \|[\tilde{y}_n] - [\tilde{y}]\|_* \leq \frac{2}{n} + \lim_{k\to\infty} \|[\tilde{x}_n] - [\tilde{x}_k]\|_*.$$

Given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that

$$\frac{2}{n} < \frac{\varepsilon}{2} \quad \text{and} \quad \|[\tilde{x}_n] - [\tilde{x}_k]\|_* < \frac{\varepsilon}{2} \qquad \forall\, n, k \geq n_0.$$

Then we obtain

$$\|[\tilde{x}_n] - [\tilde{y}]\|_* \leq \frac{2}{n} + \lim_{k\to\infty} \|[\tilde{x}_n] - [\tilde{x}_k]\|_* < \varepsilon \qquad \forall\, n, k \geq n_0.$$

Thus, $\|[\tilde{x}_n] - [\tilde{y}]\|_* \to 0$ as $n \to \infty$. ∎

Remark 1.3.2 It can be easily seen that if a normed linear space is linearly isometric with a dense subspace of a Banach space Y and also a dense subspace of another Banach space Z, then Y is linearly isometric with Z (verify!). Hence, the completion of a normed linear space as described in Theorem 1.3.16 can be thought of as *unique upto linear isometry.*

Recall that

1. c_{00} is dense in c_0 with respect to $\|\cdot\|_\infty$;

2. c_{00} is dense in ℓ^p with respect to for any p with $1 \leq p < \infty$;

Therefore, the Banach spaces c_0 and ℓ^p with $1 \leq p < \infty$ are linearly isometric with the completions of c_{00} with respect to the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively.

It can be shown that the completion of $C[a,b]$ with respect to $\|\cdot\|_p$ for $1 \leq p < \infty$ is linearly isometric with the space $L^p[a,b]$ (cf. Nair [5]). Here, by the space $L^p[a,b]$ we mean the space of all *Lebesgue measurable functions* $f : [a,b] \to \mathbb{K}$ such that

$$\int_a^b |f(t)|^p dm(t) < \infty,$$

where $m(\cdot)$ denotes the *Lebesgue measure*, and the norm on $L^p[a,b]$ is given by

$$\|f\|_p := \Big(\int_a^b |f(t)|^p dm(t)\Big)^{1/p}, \quad f \in L^p[a,b].$$

For a comprehensive treatment of Lebesgue measure and Lebesgue integral one may refer Royden [8], and for properties of L^p-spaces in the abstract setting and in the setting of topological spaces, one may refer Rudin [10]. It is to be remarked that the *Hölders inequality*

$$\int_a^b |f(t)g(t)|dm(t) \leq \|f\|_p \|g\|_q$$

and the *Minkowski inequality*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

hold for measurable functions f and g. Here, for $p = \infty$, $L^\infty[a,b]$ is the space of all functions $f :: [a,b] \to \mathbb{K}$ which are *essentially bounded*, i.e., there exists $M > 0$ such that

$$|f(x)| \leq M \quad \text{a.e. on} \quad [a,b],$$

and the norm on $L^\infty[a,b]$ is the quantity

$$\|f\|_\infty := \inf\{M > 0 : |f| \le M \text{ a.e.}\}.$$

$\Diamond$

1.4 More About Inner Product Spaces

In this section, we study more about inner product spaces. First let us consider a concept for inner product spaces which distinguish them from general normed linear spaces.

1.4.1 Orthogonality and orthonormal basis

Definition 1.4.1 Let X be an inner product space.

1. Elements x and y in an inner product space are said to be **orthogonal** to each other if $\langle x, y\rangle = 0$, and in that case, we may also write $x \perp y$.
2. For $E \subseteq X$, the set
$$E^\perp := \{x \in X : \langle x, u\rangle = 0 \,\forall\, u \in E\}$$
is called the **orthogonal compliment** of E.

$\Diamond$

- For a subset E of an inner product space X,
$$E^\perp = [\mathrm{span}\,(E)]^\perp = [\mathrm{cl}\,\mathrm{span}\,(E)]^\perp.$$

The proof of the following theorem is omitted as the result can be verified easily using the definition of the induced norm and the orthogonality.

Theorem 1.4.1 *Let X be an inner product space.*

(i) **(Parallelogram law)** *For every x, y in X,*
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(ii) **(Pythagoras theorem)** *If x, y in X are such that $\langle x, y\rangle = 0$, then*
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 1.4.2 Suppose X is an inner product space and $E \subseteq X$. Then

(i) E is said to be an **orthogonal set** if for every $x, y \in E$,

$$x \neq y \implies \langle x, y \rangle = 0,$$

(ii) E is said to be an **orthonormal set** set if it is an orthogonal set and $\|x\| = 1$ for every $x \in E$.

(iii) A sequence (u_n) in X is called an **orthonormal sequence** in X if the set $\{u_n : n \in \mathbb{N}\}$ is an orthonormal set.

$\Diamond$

- Every orthogonal set which does not contain the zero vector is linearly independent.
- If E is a basis for a linear space X, then there exists an inner product on X with respect to which E is an orthonormal set.

The proof of the following theorem is easy and hence left as an exercise.

Theorem 1.4.2 *Let X be a linear space and Y be an inner product space with inner product $\langle \cdot, \cdot \rangle_Y$. Suppose there is a bijective linear operator $T : X \to Y$. Then*

$$\langle x, u \rangle_X := \langle Tx, Tu \rangle_Y, \qquad x, u \in X,$$

defines an inner product of X.

For the next theorem, we recall the following definitions from real analysis:

1. A subset S of a metric space is said to be *dense* in Ω if $\overline{S} = \Omega$, and
2. a metric space is said to be *separable* if it has a countable dense subset.

Theorem 1.4.3 *Every orthonormal set in a separable inner product space is countable.*

Proof. Let X be a separable inner product space and D be a countable dense subset of X. Let E be an orthonormal set in X. Then for every $u, v \in E$ with $u \neq v$, we have $\|u - v\| = \sqrt{2}$. Hence, the family of all open balls $B(u, \sqrt{2})$ with $u \in E$ is a pairwise disjoint family. Since D is dense in X, each ball $B(u, \sqrt{2})$ must contain at least one point from D. Hence, E cannot be uncountable. ∎

What about the converse of the above theorem?

Way back in 1953, Dixmier [2] gave an example of a non-separable inner product space for which every orthonormal set is countable. However, the answer to the above raised question is affirmative if X is a Hilbert space, which we shall prove in a later subsection.

A typical example of a separable (infinite dimensional) Hilbert space is ℓ^2. We shall show that this space is, in fact, the only separable (infinite dimensional) Hilbert space, up to linear isometry.

Definition 1.4.3 Suppose X is an inner product space. An orthonormal set $E \subseteq X$ is said to be an **orthonormal basis** if it is a maximal orthonormal set, i.e., if $\widetilde{E}$ is an orthonormal set which contains E, then $\widetilde{E} = E$. ♢

Observe:

- An orthonormal set is an orthonormal basis if and only if its orthogonal compliment is the zero space.
- An orthonormal set which is also a basis is an orthonormal basis.
- Every orthonormal basis of a finite dimensional inner product space is a basis.

Example 1.4.1 Consider c_{00} with the ℓ^2– inner product. We may observe that $E = \{e_1, e_2, \ldots\}$ is a basis of c_{00}, and it is also an orthonormal set. Hence, E is an orthonormal basis of c_{00}.

Also, for $x \in \ell^2$, $\langle x, e_n \rangle = 0$ for all $n \in \mathbb{N}$ implies $x = 0$. Hence, E is an orthonormal basis of ℓ^2 as well. However, E is not a basis of ℓ^2. □

Question: Can an orthonormal basis of a Hilbert space be a basis?

We know that if E is a denumerable orthonormal basis of a Hilbert space X, then it cannot be a basis of X. What about if E is uncountable? The answer is given in the following theorem.

Theorem 1.4.4 *Let X be a Hilbert space and E be an orthonormal basis of X. Then E is a basis of X if and only if X is finite dimensional.*

Proof. We have observed that if X is finite dimensional, then every orthonormal basis of X is a basis.

Conversely, suppose E is a basis of X. Assume for a moment that X is infinite dimensional and $E_0 = \{u_1, u_2, \ldots\}$ is a denumerable subset of E. We know that $E_0 \neq E$. Let $X_0 = \operatorname{cl}\operatorname{span}(E_0)$. We show that E_0 is a basis of X_0, which is a contradiction, since X_0 is also a Hilbert space.

So, let $x \in X_0$. We have to show that $x \in \operatorname{span}(E_0)$. Suppose $x \notin \operatorname{span}(E_0)$. Since E is a basis of X, there exists $u_1, \ldots, u_n$ in E_0 and $v_1, \ldots, v_m$ in $E \setminus E_0$, and scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that

$$x = \alpha_1 u_1 + \ldots + \alpha_n u_n + \beta_1 v_1 + \ldots + \beta_m v_m.$$

Since $\alpha_1 u_1 + \ldots + \alpha_n u_n \in X_0$ and $\beta_1 v_1 + \ldots + \beta_m v_m \in X_0^\perp$, it follows that

$$x - (\alpha_1 u_1 + \ldots + \alpha_n u_n) = \beta_1 v_1 + \ldots + \beta_m v_m \in X_0 \cap X_0^\perp = \{0\}.$$

Thus, we have $x = \alpha_1 u_1 + \ldots + \alpha_n u_n \in \operatorname{span}(E_0)$. ∎

We know, by making use of Zorn's lemma, that every linear space has a basis. Using similar argument, it can be shown that:

- Every inner product space has an orthonormal basis.

1.4.2 Bessel's inequality

Theorem 1.4.5 (Bessel's inequality) *Let X be an inner product space and $\{u_1, \ldots, u_n\}$ be an orthonormal set in X. Then, for every $x \in X$,*

$$\sum_{i=1}^{n} |\langle x, u_i \rangle|^2 \leq \|x\|^2.$$

If X is infinite dimensional and (u_n) is an orthonormal sequence in X, then for every $x \in X$,

$$\sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2 \leq \|x\|^2.$$

Proof. Let $x \in X$. Then taking $x_n := \sum_{i=1}^n \langle x, u_i\rangle u_i$, we have

$$\|x - x_n\|^2 = \|x\|^2 + \|x_n\|^2 - \langle x, x_n\rangle - \langle x_n, x\rangle.$$

Note that

$$\|x_n\|^2 = \langle x, x_n\rangle = \langle x_n, x\rangle = \sum_{i=1}^n |\langle x, u\rangle|^2.$$

Thus,

$$\|x - x_n\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, u\rangle|^2.$$

In particular,

$$\sum_{i=1}^n |\langle x, u_i\rangle|^2 \le \|x\|^2.$$

If (u_n) is an orthonormal sequence in X, then letting n tend to ∞ in the above inequality, we obtain $\sum_{i=1}^\infty |\langle x, u_i\rangle|^2 \le \|x\|^2$. ∎

The proof of the following corollary is immediate from the last part of Theorem 1.4.5.

Corollary 1.4.6 *Suppose (u_n) is an orthonormal sequence in an inner product space X. Then, for every $x \in X$,*

$$\langle x, u_n\rangle \to 0 \quad \textit{as} \quad n \to \infty.$$

Theorem 1.4.7 *Let X be an inner product space and E be an orthonormal set in X. Then, for every $x \in X$, the set*

$$E_x := \{u \in E : \langle x, u\rangle \ne 0\}$$

is countable.

Proof. Let $x \in X$. For each $n \in \mathbb{N}$, let

$$E_{x,n} = \{u \in E : |\langle x, u\rangle| \ge 1/n\}.$$

Clearly, $E_x = \cup_{n\in\mathbb{N}} E_{x,n}$. Hence, it is enough to prove that each $E_{x,n}$ is a finite set. For fixed $n \in \mathbb{N}$, consider $u_1, \ldots, u_k$ in E_n. Then, by Bessel inequality, we have

$$\frac{k}{n^2} \le \sum_{i=1}^k |\langle x, u_i\rangle|^2 \le \|x\|^2$$

so that $k \le n^2\|x\|^2$. This shows that $E_{x,n}$ is a finite set for each $n \in \mathbb{N}$. ∎

We know from real analysis that if (a_n) is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, then for any bijection $\varphi : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^{\infty} a_{\varphi(n)}$ also converges, and

$$\sum_{n=1}^{\infty} a_{\varphi(n)} = \sum_{n=1}^{\infty} a_n. \qquad (*)$$

Thus, in view of Theorem 1.4.7, if E is an orthonormal basis in an inner product space X, then for $x \in X$, we may write

$$\sum_{u\in E} |\langle x, u\rangle|^2 \quad \text{for the sum} \quad \sum_{u\in E_x} |\langle x, u\rangle|^2,$$

where $E_x = \{u \in E : \langle x, u\rangle \neq 0\}$.

We also know from real analysis that the equality in $(*)$ above need not hold for a general convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers. However, in the setting of inner product spaces, we have the following result, which would simplify certain notations.

Proposition 1.4.8 *Let (u_n) be an orthonormal sequence in an inner product space and $x \in X$. If $\sum_{i=1}^{\infty}\langle x, u_i\rangle u_i$ converges, then for every bijection $\varphi : \mathbb{N} \to \mathbb{N}$, the series $\sum_{i=1}^{\infty}\langle x, u_{\varphi(i)}\rangle u_{\varphi(i)}$ converges, and*

$$\sum_{i=1}^{\infty}\langle x, u_{\varphi(i)}\rangle u_{\varphi(i)} = \sum_{i=1}^{\infty}\langle x, u_i\rangle u_i.$$

Proof. Suppose $\sum_{i=1}^{\infty}\langle x, u_i\rangle u_i$ converges and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection . For $n \in \mathbb{N}$, let $v_n = u_{\varphi(n)}$,

$$y_n := \sum_{i=1}^{n}\langle x, u_i\rangle u_i \quad \text{and} \quad z_n := \sum_{i=1}^{n}\langle x, v_i\rangle v_i.$$

Suppose $y_n \to y$. We have to show that $z_n \to y$.

Note that

$$\|y\|^2 = \lim_{n\to\infty} \|y_n\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\langle x, u_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle x, u_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle x, v_i\rangle|^2$$

and

$$\|y - z_n\|^2 = \|y\|^2 + \|z_n\|^2 - \langle y, z_n\rangle - \langle z_n, y\rangle.$$

Taking k large enough such that $\{v_1, \ldots, v_n\} \subseteq \{u_1, \ldots, u_k\}$, we see that

$$\langle y_k, z_n \rangle = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2 = \langle z_n, y_k \rangle.$$

Now, since $\|z_n\|^2 = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$, and

$$\langle y, z_n \rangle = \lim_{k\to\infty} \langle y_k, z_n \rangle \quad \text{and} \quad \langle z_n, y \rangle = \lim_{k\to\infty} \langle z_n, y_k \rangle,$$

we obtain

$$\begin{aligned} \|y - z_n\|^2 &= \|y\|^2 + \|z_n\|^2 - \langle y, z_n \rangle - \langle z_n, y \rangle \\ &= \|y\|^2 - \sum_{i=1}^{n} |\langle x, v_i \rangle|^2 \\ &\quad \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This completes the proof. ∎

In view of Proposition 1.4.8 and Theorem 1.4.7, if E is an orthonormal set in an inner product space X and for $x \in X$, if

$$E_x := \{u \in E : \langle x, u \rangle \neq 0\} = \{u_1, u_2, \ldots\}$$

is such that $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ converges, then we may write this sum as $\sum_{u\in E} \langle x, u \rangle u$.

Convention: For an orthonormal set E in an inner product space X and $x \in X$, if we say $\sum_{u\in E} \langle x, u \rangle u$ converges, then it means that either it is a finite sum or an infinite series which converges.

Note that the conclusion in Proposition 1.4.8 is made under the assumption of convergence of the series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$.

1.5 Hilbert Spaces

In this section we prove some results for Hilbert spaces.

Theorem 1.5.1 *Let X be a Hilbert space and E be an orthonormal set in X. Then for every $x \in X$, $\sum_{u\in E} \langle x, u \rangle u$ converges, say to $y \in X$, and $x - y \in E^{\perp}$.*

Proof. Let $x \in X$ and $E_x = \{u \in E : \langle x, u\rangle \neq 0\}$. We know that E_x is a countable set. First consider the case that E_x is a finite set, say $E_x = \{u_1, \ldots, u_n\}$. If $y = \sum_{i=1}^{n} \langle x, u_i\rangle u_i$, then we have

$$\langle x - y, u_j\rangle = 0 \quad \forall\, j \in \{1, \ldots, n\}.$$

Since $\langle x, u\rangle = 0 = \langle y, u\rangle$ for every $u \in E \setminus E_x$, we obtain

$$\langle x - y, u\rangle = 0 \quad \forall\, u \in E.$$

Hence, $x - y \in E^{\perp}$.

Next assume that E_x is denumerable, say $E_x = \{u_1, u_2, \ldots\}$. For $n \in \mathbb{N}$, let $y_n = \sum_{i=1}^{n} \langle x, u_i\rangle u_i$. Then for $n, m \in \mathbb{N}$ with $n > m$, we have

$$\|y_n - y_m\|^2 = \sum_{i=m+1}^{n} |\langle x, u_i\rangle|^2.$$

In view of Bessel's inequality, (y_n) is a Cauchy sequence in X. Since X is a Hilbert space, (y_n) converges to some $y \in X$. As in the first part, we obtain $x - y \in E^{\perp}$. ∎

1.5.1 Fourier expansion and Parseval's formula

Suppose X is a finite dimensional inner product space and $\{u_1, \ldots, u_n\}$ is an orthonormal basis of X. Then it can be easily seen that for every $x \in X$,

$$x = \sum_{i=1}^{n} \langle x, u_i\rangle u_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{n} |\langle x, u_i\rangle|^2.$$

One may ask whether such representations are possible in the setting of an infinite dimensional inner product space. We give an affirmative answer when the space is a Hilbert space.

Theorem 1.5.2 *Let X be a Hilbert space and E be an orthonormal set in X. Then, the following are equivalent.*

(i) *E is an orthonormal basis.*

(ii) **(Fourier expansion)** *For every $x \in X$, $x = \sum_{u \in E} \langle x, u\rangle u$.*

(iii) **(Parseval's formula)** *For every $x \in X$, $\|x\|^2 = \sum_{u \in E} |\langle x, u\rangle|^2$.*

Proof. (i)$\Longrightarrow$ (ii): Suppose E is an orthonormal basis and $x \in X$. By Theorem 1.5.1, $\sum_{u\in E}\langle x, u\rangle u$ converges, to say $y \in X$ and $x-y \in E^\perp$. Since E is an orthonormal basis, $E^\perp = \{0\}$ so that $y = x$.

(ii)$\Longrightarrow$ (iii): This follows from the continuity of the inner product.

(iii)$\Longrightarrow$ (i): By (iii), if $x \in X$ is such that $\langle x, u\rangle = 0$ for all $u \in E$, then $x = 0$, so that $E^\perp = \{0\}$ and hence E is an orthonormal basis. Thus, the proof is over. ∎

The following corollary is immediate from the above theorem.

Corollary 1.5.3 *If $\{u_1, u_2, \ldots\}$ is an orthonormal basis of a Hilbert space X, then for every $x \in X$,*

$$x = \sum_{n=1}^{\infty}\langle x, u_n\rangle u_n \quad \textit{and} \quad \|x\|^2 = \sum_{n=1}^{\infty}|\langle x, u_n\rangle|^2.$$

Remark 1.5.1 If E is a denumerable orthonormal basis of a (separable) Hilbert space X, say $E = \{u_n : n \in \mathbb{N}\}$, then for $x \in X$, the scalars $\langle x, u_n\rangle$ for $n \in \mathbb{N}$ are called the *Fourier coefficients* of x. ♢

1.5.2 Riesz-Fischer theorem

By Theorem 1.5.2, it is clear that if E is a denumerable orthonormal basis, say $E = \{u_n : n \in \mathbb{N}\}$ of a Hilbert space, then for every $x \in X$, the sequence $(\langle x, u_n\rangle)$ belongs to ℓ^2. In fact, the converse is also true.

Theorem 1.5.4 (Riesz-Fischer Theorem) *Suppose X is a Hilbert space with a denumerable orthonormal basis $E = \{u_n : n \in \mathbb{N}\}$. Then for every $(\alpha_n) \in \ell^2$ there exists a unique $x \in X$ such that $\alpha_n = \langle x, u_n\rangle$ for all $n \in \mathbb{N}$.*

Proof. Suppose $(\alpha_n) \in \ell^2$. For each $n \in \mathbb{N}$, let $x_n = \sum_{i=1}^{n}\alpha_i u_i$. Then, for $n > m$, we have

$$\|x_n - x_m\|^2 = \sum_{i=m+1}^{n}|\alpha_i|^2.$$

Since $(\alpha_n) \in \ell^2$, it follows that (x_n) is a Cauchy sequence in X. Since X is a Hilbert space, there exists $x \in X$ such that $x = \sum_{i=1}^{\infty}\alpha_i u_i$. Clearly, $\langle x, u_n\rangle = \alpha_n$ for all $n \in \mathbb{N}$. Orthonormality of E implies that this x is unique. ∎

In view of Theorems 1.5.2 and 1.5.4, we have the following theorem.

Theorem 1.5.5 *A separable Hilbert space is linearly isometric with either* $\mathbb{K}^N$ *for some* $N \in \mathbb{N}$ *or* ℓ^2 *according as it is finite dimensional or infinite dimensional.*

Proof. Let $E := \{u_n : n \in \Lambda\}$ be a countable orthonormal basis of X, where $\Lambda = \{1, \ldots, N\}$ if $\dim(X) = N < \infty$ and $\Lambda = \mathbb{N}$ if $\dim(X) = \infty$. Now, by Theorems 1.5.2 and 1.5.4, we have the following:

(i) Suppose X is finite dimensional with $\dim(X) = N$. Then, the map

$$x \mapsto (\langle x, u_1 \rangle, \ldots, \langle x, u_N \rangle)$$

is a linear isometry from X onto $\mathbb{K}^N$.

(ii) Suppose X is infinite dimensional. Then the map

$$x \mapsto (\langle x, u_1 \rangle, \langle x, u_2 \rangle, \ldots)$$

is a linear isometry from X onto ℓ^2. ∎

1.5.3 Projection theorem

Now, we prove an important theorem in Hilbert space theory, the so called *projection theorem.*

Theorem 1.5.6 (Projection theorem) *Let* X *be an inner product space and* X_0 *be a complete subspace of* X*. Then*

$$X = X_0 + X_0^{\perp} \quad \textit{and} \quad X_0^{\perp\perp} = X_0.$$

In particular, every $x \in X$ *can be uniquely represented as*

$$x = y + z \quad \textit{with} \quad y \in X_0,\ z \in X_0^{\perp}.$$

Proof. Let E be an orthonormal basis of X_0 and $x \in X$. By Theorem 1.5.1, $\sum_{u \in E} \langle x, u \rangle u$ converges, to say $y \in X_0$ and $x - y \in E^{\perp}$. Sine X_0 is complete and E is an orthonormal basis of X_0, by Theorem 1.5.2, any $z \in X_0$ can be written as $z = \sum_{u \in E} \langle z, u \rangle u$. Hence, we obtain

$$x - y \in X_0^{\perp}.$$

Thus, every $x \in X$ can be represented as $x = y + z$ with $y \in X_0$ and $z \in X_0^{\perp}$, and hence $X = X_0 + X_0^{\perp}$.

It remains to show that $X_0^{\perp\perp} = X_0$. It is easy to see the inclusion $X_0 \subseteq X_0^{\perp\perp}$. To see the reverse inclusion, let $x \in X_0^{\perp\perp}$ and let $u \in X_0$ and $v \in X_0^{\perp}$ be such that $x = u + v$. Then we have

$$0 = \langle x, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \langle v, v\rangle.$$

Thus, $v = 0$ so that $x = u \in X_0$. Therefore, we obtain $X_0^{\perp\perp} \subseteq X_0$. The last part of the theorem is obvious. ∎

In most of the books on Functional Analysis, projection theorem is written in the following way, which is an immediate consequence of Theorem 1.5.6

Corollary 1.5.7 (Projection theorem) *Let X be a Hilbert space and X_0 be a closed subspace of X. Then*

$$X = X_0 + X_0^{\perp} \quad \textit{and} \quad X_0^{\perp\perp} = X_0.$$

Here are two consequences of projection theorem.

Theorem 1.5.8 *Suppose X is a Hilbert space and E is an orthonormal basis of X. Then* span (E) *is dense in X. In particular, X is separable if and only if E is countable.*

Proof. Let $X_0 = \mathrm{cl\,span}\,(E)$. Since $E^{\perp} = \{0\}$, it follows that $X_0^{\perp} = \{0\}$. Hence, by Projection theorem, $X = X_0$, showing that span (E) is dense in X.

By Theorem 1.4.3 we know that if X is separable, then E is countable. Next, suppose that E is countable. Clearly, countability of E implies that span (E) is separable, and hence $X = \mathrm{cl\,span}\,(E) = X_0$ is also separable. ∎

Theorem 1.5.9 (Best approximation) *Let X be an inner product space and X_0 be a complete subspace of X. Then for every $x \in X$, there exists a unique $y \in X_0$ such that*

$$\|x - y\| = \mathrm{dist}\,(x, X_0).$$

Proof. By Theorem 1.5.6, every $x \in X$ can be written as uniquely as $x = y + z$ with $y \in X_0$ and $z \in X_0^{\perp}$. Clearly, $\|x - y\| \geq \mathrm{dist}\,(x, X_0)$. By Pythagoras theorem, for any $w \in X_0$,

$$\|x - w\|^2 = \|(x - y) + (y - w)\|^2 = \|x - y\|^2 + \|y - w\|^2 \geq \|x - y\|^2.$$

Since this true for any $w \in X_0$, we obtain dist $(x, X_0) \geq \|x - y\|$. To see the uniqueness, suppose $y_1 \in X_0$ be such that

$$\|x - y_1\| = \text{dist}\,(x, X_0).$$

Then, again by Pythagoras theorem,

$$\|x - y_1\|^2 = \|(x - y) + (y - y_1)\|^2 = \|x - y\|^2 + \|y - y_1\|^2.$$

Sine $\|x - y_1\| = \|x - y\|$, it follows that $y_1 = y$. This completes the proof. ∎

Definition 1.5.1 A linear operator $P : X \to X$ on a linear space X is said to be a **projection operator** onto a subspace X_0 if

$$R(P) = X_0 \quad \text{and} \quad Px = x \qquad \forall\, x \in X_0.$$

$\Diamond$

It can be seen that linear operator $P : X \to X$ is a projection operator if and only if $P^2 = P$, and in that case,

$$R(P) = N(I - P), \qquad R(I - P) = N(P).$$

Definition 1.5.2 A projection operator $P : X \to X$ on an inner product space X is said to be an **orthogonal projection** if

$$R(P) \perp N(P).$$

$\Diamond$

Theorem 1.5.10 *Let X be an inner product space and $P : X \to X$ be a projection operator. Then P is an orthogonal projection if and only if*

$$\langle Px, y\rangle = \langle x, Py\rangle \quad \forall\, x, y \in X.$$

Proof. Suppose P is an orthogonal projection. Let $x, y \in X$, and let $x_1, y_1 \in R(P)$ and $x_2, y_2 \in N(P)$ such that

$$x = x_1 + x_2, \quad y = y_1 + y_2.$$

Since $Px = x_1$, $Py = y_1$ and $\langle x_i, y_j\rangle = 0$ for $i \neq j$, we have

$$\langle Px, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle,$$

$$\langle x, Py\rangle = \langle x_1 + y_1, y_1\rangle = \langle x_1, y_1\rangle.$$

Thus, $\langle Px, y\rangle = \langle x, Py\rangle$.

To prove the converse, suppose $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in X$. Let $x \in N(P)$ and $y \in R(P)$. Then

$$\langle x, y\rangle = \langle x, Py\rangle = \langle Px, y\rangle = 0,$$

since $Px = 0$. Thus, $N(P) \perp R(P)$. ∎

Theorem 1.5.11 *Let X_0 be a complete subspace of an inner product space X. Then there exists an orthogonal projection onto X_0.*

Proof. By Theorem 1.5.6, every $x \in X$ can be written uniquely as $x = y + z$ with $y \in X_0$ and $z \in X_0^{\perp}$. Define $Px = y$. Then it is easily seen that $P : X \to X$ is an orthogonal projection onto X_0. ∎

1.6 Problems

1. Prove the following:

   (a) The zero vector in condition 3 in the Definition 1.1.1 is unique.

   (b) The vector $\tilde{x}$ in condition 4 in the Definition 1.1.1, corresponding to a vector $x \in X$, is unique.

   (c) If $\alpha, \beta \in \mathbb{K}$ and x is a nonzero vector in X such that $\alpha x = \beta x$, then $\alpha = \beta$.

2. Prove that if $f : \mathbb{K}^n \to \mathbb{K}$ is a linear functional, then there exist $(a_1, \ldots, a_n) \in \mathbb{K}^n$ such that

$$f(x) = a_1 x_1 + \ldots a_n x_n, \qquad x = (x_1, \ldots, x_n) \in \mathbb{K}^n.$$

3. Prove that a bounded metric on a nonzero linear space does not induce a norm.

4. Let X be a normed linear space. Prove the following:

   (a) For every $x_0 \in X$, the function $\alpha \mapsto \alpha x_0$ is continuous from $\mathbb{K}$ to X.

   (b) For every $x_0 \in X$ and $\alpha \in \mathbb{K}$, the function $x \mapsto x + \alpha x_0$ is continuous from X to X.

5. Let X be a normed linear space and let (x_n) be a sequence in X. Prove the following:

   (a) If (α_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n$ converges and if
   $$\|x_{n+1} - x_n\| \le \alpha_n \quad \forall\, n \in \mathbb{N},$$
   then (x_n) is a Cauchy sequence in X.

   (b) If (x_n) is a Cauchy sequence and if it has a subsequence which converge to $x \in X$, then (x_n) also converges to x.

   (c) If (x_n) is a Cauchy sequence and if (α_n) be a sequence of positive real numbers such that $\alpha_n \to 0$, then there exists a strictly increasing sequence (k_n) in $\mathbb{N}$ such that
   $$\|x_{k_{n+1}} - x_{k_n}\| \le \alpha_n \quad \forall\, n \in \mathbb{N}.$$

6. Let X be a linear space and $\nu : X \to \mathbb{R}$ be such that $\nu(x) \ne 0$ whenever $x \ne 0$ and for every $x, y \in X$ and $\alpha \in \mathbb{K}$,
   $$\nu(x + y) \le \nu(x) + \nu(y), \qquad \nu(\alpha x) = \alpha \nu(x).$$
   Prove that $\nu(0) = 0$ and $\nu(x) \ge 0$ for all $x \in X$.

7. If $\|\cdot\|$ is a norm on $\mathbb{K}$ such that $\|1\| = 1$, then prove that $\|x\| = |x|$ for all $x \in \mathbb{K}$.

8. Let X be a normed linear space with norm $\|\cdot\|$. Define
   $$d(x, y) = \min\{1, \|x - y\|\}, \qquad x, y \in X.$$
   Prove that d is a metric on X which is not induced by any norm on X.

9. Prove that
   $$c_{00} \subseteq \ell^1 \subseteq \ell^p \subseteq \ell^r \subseteq c_0 \subseteq c \subseteq \ell^\infty$$
   for every $p, r \in (1, \infty)$ with $p < r$. Also, prove that the inclusions are strict.

10. Prove that for $0 < p < 1$, $\|x\|_p := \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p}$ does not define a norm on $\{x \in \mathcal{F}(\mathbb{N}) : \sum_{i=1}^{\infty} |x(i)|^p < \infty\}$.

11. Verify the following:

    (a) For any p and r with $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ on $\mathbb{K}^n$ are equivalent.

    (b) For $1 \leq p < \infty$, the norm $\|\cdot\|_p$ on $C[a,b]$ is weaker than the supremum norm $\|\cdot\|_\infty$.

    (c) For $1 \leq p < r \leq \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ on $C[a,b]$ are not equivalent.

    (d) For $1 \leq p < r \leq \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ on ℓ^p are not equivalent.

12. Let X be a normed linear space and (x_n) be a sequence in X. Prove the following.

    (i) If (x_n) is Cauchy sequence, then there exists a subsequence (x_{k_n}) of (x_n) such that the series $\sum_{n=1}^{\infty}(x_{k_{n+1}} - x_{k_n})$ is absolutely convergent.

    (ii) If the series $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ is absolutely convergent, then (x_n) is a Cauchy sequence.

13. Prove that $C[a,b]$ is a Banach space with respect to $\|\cdot\|_p$ if and only if $p = \infty$.

14. Prove that ℓ^1 is not a closed subspace of ℓ^∞ and ℓ^p is not a Banach space with respect to the norm $\|\cdot\|_\infty$.

15. Prove that for $1 \leq p < r \leq \infty$, ℓ^p is not a closed subspace of ℓ^r; in particular, ℓ^p is not a Banach space with respect to the norm $\|\cdot\|_r$.

16. Prove that

    (a) c, the space of all convergent scalar sequences, is a closed subspace of ℓ^∞.

    (b) c_0, the space of all convergent scalar sequences having limit 0, is a closed subspace of c_0.

17. Justify the following statements:

    (a) The space c_{00} is not a Banach space with respect to any norm.

(b) The space $\mathcal{P}$ of all polynomials is not a Banach space with respect to any norm.

(c) Every infinite dimensional space has a subspace which is not a Banach space with respect to any norm.

(d) A Banach space is finite dimensional if and only if every subspace of it is closed.

18. Let $C_0(\mathbb{R})$ be the set of a continuous functions $x : \mathbb{R} \to \mathbb{K}$ such that $|x(t)| \to 0$ as $|t| \to \infty$. Prove that $C_0(\mathbb{R})$ is a closed subspace of $B(\mathbb{R})$.

19. Let $C_c(\mathbb{R})$ be the set of a continuous functions $x : \mathbb{R} \to \mathbb{K}$ such that $\operatorname{cl}\{t \in \mathbb{K} : x(t) \neq 0\}$ is compact in $\mathbb{K}$. Prove that $C_c(\mathbb{R})$ is a dense subspace of $C_0(\mathbb{R})$ with respect to $\|\cdot\|_\infty$.

20. If $p(\cdot)$ is a semi-norm on a linear space X, then $p(x) \geq 0$ for all $x \in X$ and $p(0) = 0$.

21. Show that the function $p(\cdot)$ in each of the following is a semi-norm, but not a norm.

   (a) Let $X = C[a, b]$ and for $\tau \in [a, b]$, let $p(x) = |x(\tau)|$, $x \in X$.

   (b) Let $X = \mathcal{R}([a, b], \mathbb{R})$, the space (over $\mathbb{R}$) of all real valued Riemann integrable functions, and let $p(x) = \int_a^b |x(t)|dt$, $x \in X$.

   (c) Let $X = \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ with $\det(A) = 0$ and $p(x) = \|Ax\|_2$, $x \in X$. (Here, $x \in \mathbb{R}^n$ is considered as a column vector.)

22. For a subset E of an inner product space X, prove that

$$E^\perp = [\operatorname{span}(E)]^\perp = [\operatorname{cl}\operatorname{span}(E)]^\perp.$$

23. Prove that if D is a dense subset of an inner product space X and if $x \in X$ is such that $\langle x, u\rangle = 0$ for all $u \in D$, then $x = 0$.

24. Verify the functions $\langle\cdot,\cdot\rangle$ defined in (i)-(iii) in Example 1.2.9 are inner product.

25. Prove Theorem 1.4.1(i) and (ii).

26. Prove the following:

(a) Every orthonormal set is linearly independent,

(b) An orthonormal set E is an orthonormal basis if and only if $E^{\perp} = \{0\}$,

(c) An orthonormal set which is also a basis is an orthonormal basis.

27. Prove Corollary 1.4.6.

28. If E is an orthonormal set in an inner product space, $x \in X$ and $r > 0$, then prove that $\{u \in E : |\langle x, u\rangle| \geq r\}$ is a finite set.

29. Let X be a Hilbert space and X_0 is a closed subspace of X. Prove the following:

    (a) For every $x \in X$, there exists a unique pair $(y, z) \in X_0 \times X_0^{\perp}$ such that $x = y + z$.

    (b) Consider the map $P : X \to X$ defined by $Px = y$, where y is obtained as in (a). Then P is an orthogonal projection.

30. Let X be a Hilbert space and E be a countable orthonormal subset of X. Define $P : X \to X$ by

$$Px = \sum_{u \in E} \langle x, u\rangle u, \qquad x \in X.$$

Prove that P is an orthogonal projection.

31. Suppose X is an infinite dimensional (separable) Hilbert space and $E = \{u_n : n \in \mathbb{N}\}$ is an orthonormal basis of X. Let $P_n : X \to X$ be defined by

$$P_n x = \sum_{i=1}^{n} \langle x, u_i\rangle u_i, \qquad x \in X.$$

Then, prove that

$$\|x - P_n x\| = \operatorname{dist}(x, R(P_n)) \qquad \forall\, x \in X$$

and for every $x \in X$,

$$\|x - P_n x\|^2 = \sum_{i=n}^{\infty} |\langle x, u_i\rangle|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

32. Show that a linear operator $P : X \to X$ is a projection operator if and only if $P^2 = P$, and in that case,

$$R(P) = N(I - P), \qquad R(I - P) = N(P).$$

33. Let X_0 be a complete subspace of an inner product space X or a closed subspace of a Hilbert space X. If P is the orthogonal projection from X onto X_0, then prove that, for every $x \in X$, Px is the unique best approximation of x from X_0.

34. Prove that if X is an inner product space and $P : X \to X$ is a projection operator, then P is an orthogonal projection if and only if

$$\langle Px, y \rangle = \langle x, Py \rangle \quad \forall\, x, y \in X.$$