2

Operators

2.1 Bounded Linear Operators

Recall from real analysis that if J is a subset of $\mathbb{R}$ and if f is a real valued function defined on J, then

(i) f continuous at a point $t_0 \in J$ does not imply that it is continuous at another point $t_1 \in J$;

(ii) f continuous at every point $t \in J$ does not imply that it is uniformly continuous on J;

(iii) f uniformly continuous on J does not imply that it is Lipschitz continuous on J.

However, we prove below that for a linear operator between normed linear spaces, *Lipschitz continuity, uniform continuity, continuity, and continuity at a point* are all equivalent.

First recall that a *linear operator* or linear transformation between linear spaces X and Y is a function $A : X \to Y$ satisfying the conditions

$$A(x + y) = A(x) + A(y) \quad \text{and} \quad A(\alpha x) = \alpha A(x)$$

for all $x, y \in X$ and $\alpha \in \mathbb{K}$.

Theorem 2.1.1 *Let X, Y be normed linear spaces and $A : X \to Y$ be a linear operator. Then the following are equivalent.*

(i) *A is continuous at the point 0.*

(ii) *There exists $c > 0$ such that $\|Ax\| \leq c\|x\|$ for all $x \in X$.*

(iii) *A is uniformly continuous on X.*

Proof. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) are obvious. Hence, it is enough to prove (i) $\Longrightarrow$ (ii).

Assume that (i) holds. Since $A(0) = 0$, there exists $\delta > 0$ such that

$$\|x\| < \delta \Longrightarrow \|Ax\| < 1.$$

Hence, for every $x \neq 0$, since the vector $\delta x/2\|x\|$ is of norm less than δ, we have

$$\|A\Big(\frac{\delta x}{2\|x\|}\Big)\| < 1,$$

so that

$$\|Ax\| \leq \frac{2}{\delta}\|x\| \quad \forall\, x \in X$$

Thus, (i) $\Longrightarrow$ (ii). ∎

Continuity of a linear operator $A : X \to Y$ is also equivalent to the following:

(a) The image of every bounded subset of X is bounded in Y.

(b) The set $\{\|Ax\| : \|x\| = 1\}$ is bounded.

In view of the characterization (a) above for a continuous linear operator, we have the following definition.

Definition 2.1.1 A continuous linear operator is also called a **bounded linear operator**. ♢

2.1.1 Space of bounded linear operators

Throughout this chapter, when we say that $A : X \to Y$ is a bounded linear operator, it is assumed that X and Y are normed linear spaces.

Notation 2.1.1 The set of all bounded linear operators from X to Y is denoted by $\mathcal{B}(X, Y)$. ♢

Thus,

$$A \in \mathcal{B}(X, Y) \iff \exists\, c > 0 \text{ such that } \|Ax\| \leq c\|x\| \quad \forall\, x \in X.$$

Theorem 2.1.2 *Let X, Y be normed linear spaces. Then $\mathcal{B}(X, Y)$ is a linear space, and the function $\nu : \mathcal{B}(X, Y) \to \mathbb{R}$ defined by*

$$\nu(A) := \inf\{c > 0 : \|Ax\| \leq c\|x\| \;\forall\, x \in X\}, \quad A \in \mathcal{B}(X, Y),$$

is a norm on $\mathcal{B}(X, Y)$.

Proof. Clearly $\mathcal{B}(X,Y)$ is a subset of the linear space $\mathcal{L}(X,Y)$ of all linear operators from X to Y. We observe that for $A \in \mathcal{B}(X,Y)$, we have

$$\nu(A) = 0 \iff A = 0.$$

and

$$\|Ax\| \le \nu(A)\|x\| \quad \forall\, x \in X.$$

Thus, for A, B in $\mathcal{B}(X,Y)$,

$$\|(A+B)x\| \le (\nu(A) + \nu(B))\|x\|, \quad \forall\, x \in X,$$

$$\|(\alpha A)(x)\| = |\alpha|\, \|Ax\| \le |\alpha|\nu(A)\|x\| \quad \forall\, x \in X.$$

Therefore $A+B$, $\alpha A \in \mathcal{B}(X,Y)$ and

$$\nu(A+B) \le \nu(A) + \nu(B), \quad \nu(\alpha A) \le |\alpha|\nu(A).$$

In particular, $\mathcal{B}(X,Y)$ is a subspace of the space $\mathcal{L}(X,Y)$. Further, the equality $\|(\alpha A)(x)\| = |\alpha|\, \|Ax\|$ for all $x \in X$ also shows that $|\alpha|\nu(A) \le \nu(\alpha A)$ so that

$$\nu(\alpha A) = |\alpha|\nu(A).$$

Thus, we have also shown that ν is a norm on $\mathcal{B}(X,Y)$. ∎

Convertion: Hereafter, the norm on the space $\mathcal{B}(X,Y)$ will be the one given in Theorem 2.1.2, and it will be denoted by $\|A\|$.

Remark 2.1.1 If $c > 0$ is such that $\|Ax\| \le c\|x\|$ for all $x \in X$, then

$$\|A\| \le c.$$

If in addition, there exists $x_0 \ne 0$ in X such that $\|Ax_0\| = c\|x_0\|$, then we also have $c \le \|A\|$ so that we obtain $\|A\| = c$. This observation will help us computing the norms of certain operators. ◊

- For $A \in \mathcal{B}(X,Y)$, the quantities

$$\alpha_A := \sup\{\|Ax\| : \|x\| \le 1\},$$

$$\beta_A := \sup\{\|Ax\| : \|x\| = 1\},$$

$$\gamma_A := \sup\left\{\frac{\|Ax\|}{\|x\|} : x \ne 0\right\}$$

  are finite and are equal to $\|A\|$.

Definition 2.1.2 We use the notation $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$ and X' for $\mathcal{B}(X, \mathbb{K})$.

1. The space X' is called the **dual space** or simply the **dual** of X and its elements are called **continuous linear functionals** or **bounded linear functionals**. Continuous linear functionals are usually denoted by small scale letters f, g, etc.

2. An operator in $\mathcal{B}(X)$ is called a a **bounded linear operator on** X.

$\Diamond$

Theorem 2.1.3 *If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space. In particular, for every normed linear space X, X' is a Banach space.*

Proof. Suppose Y is a Banach space. We have to show that every Cauchy sequence of operators in $\mathcal{B}(X, Y)$ converges to an operator in $\mathcal{B}(X, Y)$. So, let (A_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$ and $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that

$$\|A_n - A_m\| < \varepsilon \quad \forall\, n, m \geq N.$$

Hence, for any $x \in X$, we have

$$\|(A_n - A_m)x\| \leq \|A_n - A_m\|\|x\| < \varepsilon\|x\| \quad \forall\, n, m \geq N.$$

Thus, for each $x \in X$, $(A_n x)$ is a Cauchy sequence in Y. Since Y is a Banach space, $(A_n x)$ converges in Y. Let $A : X \to Y$ be defined by

$$Ax := \lim_{n\to\infty} A_n x, \quad x \in X.$$

It can be easily seen that A is a linear operator. Also, since (A_n) is a Cauchy sequence, it is bounded. Let $M > 0$ be such that $\|A_n\| \leq M$ for all $n \in \mathbb{N}$. Hence,

$$\|Ax\| = \lim_{n\to\infty} \|A_n x\| \leq M\|x\| \quad \forall x \in X.$$

Thus, $A \in \mathcal{B}(X, Y)$. Further, we have

$$\|A_n x - Ax\| = \lim_{m\to\infty} \|(A_n - A_m)x\| \leq \varepsilon\|x\| \quad \forall\, x \in X,\, n \geq N.$$

Thus, $\|A_n - A\| \leq \varepsilon$ for all $n \geq N$, showing that (A_n) converges to A in $\mathcal{B}(X, Y)$. ∎

Remark 2.1.2 We shall prove in the next chapter, as a consequence of a theorem called *Hahn-Banach extension theorem*, that the converse of Theorem 2.1.3 is also true. ◊

The following theorem gives a class of examples of bounded operators.

Theorem 2.1.4 *Let X and Y normed linear spaces and $A : X \to Y$ be a linear operator. If* $\dim(X) < \infty$*, then* $A \in \mathcal{B}(X, Y)$.

Proof. Let $\dim(X) = n$ and $E = \{u_1, \ldots, u_k\}$ be an ordered basis of X. For $x = \sum_{i=1}^{k} \alpha_i u_i$ in X, let

$$\|x\|_E := \max\{|\alpha_i| : i = 1, \ldots, k\}.$$

We know that $\|\cdot\|_E$ is a norm on X which is equivalent to the original norm on X. Thus, there exists $c_0 > 0$ such that $\|x\|_E \le c_0\|x\|$ for all $x \in X$. Hence, for all $x \in X$,

$$\|Ax\| \le \sum_{i=1}^{k} |\alpha_i|\, \|Au_i\| \le \|x\|_E \sum_{i=1}^{k} \|Au_i\| = c\|x\|,$$

where $c = c_0 \sum_{i=1}^{k} \|Au_i\|$. ∎

A natural question is whether the assumption $\dim(X) < \infty$ in the above theorem can be dropped or can be replaced by $\dim(Y) < \infty$. The following example shows that the answer is in negative.

Example 2.1.1 (A discontinuous linear functional) Let X be the space c_{00} with $\|\cdot\|_\infty$ and let $f : c_{00} \to \mathbb{K}$ be defined by

$$f(x) = \sum_{j=1}^{\infty} x(j), \quad x \in c_{00}.$$

Then f is a linear functional on X. But, $f \notin X'$. To see this, let

$$x_n(i) = \begin{cases} 1, & j \le n, \\ 0, & j > n \end{cases}$$

for $n \in \mathbb{N}$. The we see that $x_n \in c_{00}$, $\|x_n\|_\infty = 1$ and $f(x_n) = n$ for all $n \in \mathbb{N}$. Thus, (x_n) is a bounded sequence whose image is not a bounded sequence. □

The following corollary is immediate from Theorem 2.1.4 by observing that the inverse of a linear operator is a linear operator.

Corollary 2.1.5 *Any two finite dimensional normed linear spaces of the same dimension are linearly homeomorphic.*

2.1.2 Examples of bounded linear operators

Now, let us give some examples of bounded linear operators whose domains are infinite dimensional spaces.

Example 2.1.2 Let (λ_n) be a bounded sequence of scalars and $A : \ell^p \to \ell^p$ be defined by

$$(Ax)(i) = \lambda_i x(i), \quad i \in \mathbb{N}.$$

Let $\beta := \sup_{n\in\mathbb{N}} |\lambda_n|$. Then we obtain

$$\|Ax\|_p \leq \beta \|x\|_p \quad \forall\, x \in \ell^p$$

so that A is a bounded linear operator and $\|A\| \leq \beta$. Also, we have

$$|\lambda_n| = \|\lambda_n e_n\|_p = \|Ae_n\|_p \leq \|A\| \quad \forall\, n \in \mathbb{N}$$

so that $\beta \leq \|A\|$. Thus, we have proved that $\|A\| = \beta$. □

Example 2.1.3 Let $X = C[a, b]$ with $\|\cdot\|_\infty$. For $u \in C[a, b]$, let

$$(A_u x)(t) = u(t)x(t), \quad x \in C[a, b], \quad t \in [a, b].$$

Then we have

$$\|A_u x\|_\infty \leq \|u\|_\infty \|x\|_\infty \quad \forall x \in C[a, b]$$

so that $A \in \mathcal{B}(X)$ and $\|A_u\| \leq \|u\|_\infty$. Further, if $x_0(t) = 1$ for all $t \in [a, b]$, then we have

$$|u(t)| = |(A_u x_0)(t)| \leq \|A_u\| \quad \forall\, t \in [a, b]$$

so that $\|u\|_\infty \leq \|A_u\|$. Thus, we have proved that $\|A_u\| = \|u\|_\infty$.

Also, the function $T : X \to \mathcal{B}(X)$ defined by

$$T(u) = A_u, \quad u \in X,$$

is a linear operator. Note also that

$$\|T(u)\| = \|A_u\| = \|u\|, \quad u \in X,$$

so that T is a linear isometry. Thus, X can be viewed as the subspace

$$R(T) = \{A_u : u \in C[a,b]\}$$

of the space $\mathcal{B}(X)$. $\square$

Example 2.1.4 Let $X = C[a,b]$ with $\|\cdot\|_\infty$.

(i) Let

$$(Ax)(s) = \int_a^s x(t)\,dt, \quad x \in C[a,b], \quad s \in [a,b].$$

Then we see that $Ax \in C[a,b]$ for every $x \in C[a,b]$ and A is a linear operator on X. Further, we have

$$|(Ax)(s)| \le \int_a^s |x(t)|\,dt \le (b-a)\|x\|_\infty \quad \forall\, x \in C[a,b],\ s \in [a,b].$$

Hence, we have $\|Ax\|_\infty \le (b-a)\|x\|_\infty$ for all $x \in C[a,b]$, and consequently, $A \in \mathcal{B}(X)$ and $\|A\| \le b-a$. Also, since x_0 defined by $x_0(t) = 1$ for all $t \in [a,b]$ satisfies $\|Ax_0\|_\infty = (b-a)\|x_0\|_\infty$, we have $\|A\| = b-a$.

(ii) Let

$$f(x) = \int_a^b x(t)\,dt, \quad x \in C[a,b].$$

Then it can be seen (Verify) that $f \in X'$ and $\|f\| = b-a$. $\square$

Example 2.1.5 Let $X = \ell^2$, and $a_{ij} \in \mathbb{K}$ be such that

$$\beta := \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| < \infty \quad \text{and} \quad \gamma := \sup_{j\in\mathbb{N}} \sum_{i=1}^\infty |a_{ij}| < \infty.$$

We show that

$$Ax = \sum_{i=1}^\infty \Big(\sum_{j=1}^\infty a_{ij}x(j)\Big)e_i, \quad x \in \ell^2,$$

defines a bounded linear operator from ℓ^2 to itself and $\|A\| \le \sqrt{\beta\gamma}$.

Let $x \in X$. Then for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{j=1}^\infty |a_{ij}x(j)| &= \sum_{j=1}^\infty |a_{ij}|^{1/2}|a_{ij}|^{1/2}|x(j)| \\ &\le \Big(\sum_{j=1}^\infty |a_{ij}|\Big)^{1/2}\Big(\sum_{j=1}^\infty |a_{ij}||x(j)|^2\Big)^{1/2}.\end{aligned}$$

Thus,

$$\Big(\sum_{j=1}^{\infty}|a_{ij}x(j)|\Big)^2 \leq \beta\sum_{j=1}^{\infty}|a_{ij}||x(j)|^2,$$

and

$$\begin{aligned}\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|a_{ij}x(j)|\Big)^2 &\leq \beta\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}||x(j)|^2 \\ &= \beta\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|a_{ij}|\Big)|x(j)|^2 \\ &\leq \beta\gamma\|x\|_2^2.\end{aligned}$$

Hence, for each $x \in \ell^2$ and $i \in \mathbb{N}$,

$$(Ax)(i) := \sum_{j=1}^{\infty} a_{ij}x(j)$$

is well-defined, $Ax \in \ell^2$ and $\|Ax\|_2 \leq \sqrt{\beta\gamma}\|x\|_2$. Thus, $A : \ell^2 \to \ell^2$ is a a bounded operator and $\|A\| \leq \sqrt{\beta\gamma}$.

Taking $a_{ij} = \lambda_i\delta_{ij}$ for $i, j \in \mathbb{N}$, we recover the Example 2.1.2. □

Example 2.1.6 Let $k(\cdot,\cdot) \in C([a,b]\times[a,b])$. For $x \in C[a,b]$, let

$$(Ax)(s) = \int_a^b k(s,t)x(t)dt, \quad s \in [a,b].$$

We see that $Ax \in C[a,b]$ for every $x \in C[a,b]$.

(i) Let $X = C[a,b]$ with $\|\cdot\|_\infty$. Let $x \in C[a,b]$. We have

$$|(Ax)(s)| \leq \int_a^b |k(s,t)|\,|x(t)|dt \leq \|x\|_\infty\Big(\int_a^b |k(s,t)|dt\Big).$$

Thus,

$$\|Ax\|_\infty \leq \beta\|x\|_\infty, \quad \beta := \sup_{a\leq s\leq b}\int_a^b |k(s,t)|dt.$$

Therefore, $A \in \mathcal{B}(X)$ and $\|A\| \leq \beta$. In fact, it is also known that $\|A\| = \beta$ (cf. Nair [5]).

(ii) Let $X = C[a,b]$ with $\|\cdot\|_2$. Let $x \in C[a,b]$. We have

$$|(Ax)(s)| \leq \int_a^b |k(s,t)|\,|x(t)|dt \leq \|x\|_2\Big(\int_a^b |k(s,t)|^2dt\Big)^{1/2}.$$

so that

$$\|Ax\|_2^2 = \int_a^b |(Ax)(s)|^2 ds \le \Big(\int_a^b \int_a^b |k(s,t)|^2 dt\Big)\|x\|_2^2.$$

Thus, $A \in \mathcal{B}(X)$ and $\|A\| \le (\int_a^b \int_a^b |k(s,t)|^2 dt)^{1/2}$. □

Example 2.1.7 Consider the linear operator $A : C^1[0,1] \to C[0,1]$ defined by

$$(Ax)(t) = x'(t), \quad x \in C^1[0,1], \quad t \in [0,1].$$

Taking

$$x_n(t) = \frac{t^n}{n+1}, \quad n \in \mathbb{N}, \quad t \in [0,1],$$

we have

$$\|x_n\|_\infty = \frac{1}{n+1} \quad \text{and} \quad \|Ax_n\|_\infty = \frac{n}{n+1}.$$

Thus, with respect to $\|x_n\|_\infty \to 0$, but $\|Ax_n\|_\infty \not\to 0$. Hence, with respect to the norm $\|\cdot\|_\infty$ on both the spaces $C^1[0,1]$ and $C[0,1]$, A is not a bounded operator. However, if we take the norm

$$\|x\|_* := \|x\|_\infty + \|x'\|_\infty, \quad x \in C^1[0,1]$$

on $C^1[0,1]$, we have

$$\|Ax\|_\infty = \|x'\|_\infty \le \|x\|_* \quad \forall\, x \in C^1[0,1].$$

Thus, taking

$$X = C^1[0,1] \text{ with } \|\cdot\|_* \text{ and } Y = C[0,1] \text{ with } \|\cdot\|_\infty,$$

we obtain

$$A \in \mathcal{B}(X,Y) \quad \text{and} \quad \|A\| \le 1.$$

Also, with x_n as above, we have $\|x_n\|_* = 1$ and

$$\frac{n}{n+1} = \|Ax_n\|_\infty \le \|A\|\|x_n\|_* = \|A\| \quad \forall n \in \mathbb{N}$$

so that we obtain $\|A\| = 1$. □

Example 2.1.8 Let X be an inner product space and $P : X \to X$ be a nonzero orthogonal projection. Then for every $x \in X$, since

$$Px \in R(P) \quad \text{and} \quad (I - P)x \in N(P) = R(P)^{\perp},$$

we have

$$\|x\|^2 = \|Px + (I - P)x\|^2 = \|Px\|^2 + \|(I - P)x\|^2 \geq \|Px\|^2.$$

Hence,

$$\|Px\| \leq \|x\| \quad \forall\, x \in X,$$

showing that $P \in \mathcal{B}(X)$ and $\|P\| \leq 1$. Since P is nonzero, there exists a nonzero $x \in X$ such that $Px = x$ so that

$$\|x\| = \|Px\| \leq \|P\|\, \|x\|,$$

and hence, we also have $\|P\| \leq 1$. Thus, $\|P\| = 1$. □

Example 2.1.9 Let X be an infinite dimensional Hilbert space and (u_n) be an orthonormal sequence in X. Let (λ_n) be a bounded sequence of scalars. For $x \in X$, define

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n.$$

The above operator $A : X \to X$ is well defined. Indeed, if M is a bound for $(|\lambda_n|)$, then for every $x \in X$,

$$\sum_{n=1}^{\infty} |\lambda_n|^2\, |\langle x, u_n \rangle|^2 \leq M^2 \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq M\|x\|^2,$$

so that by Riesz-Fischer theorem (Theorem 1.5.4), the series $\sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n$ is convergent. It can be easily seen that A is a linear operator from X to itself. Further, we have

$$\|Ax\|^2 \leq M^2 \|x\|^2 \quad \forall\, x \in X.$$

Hence, $A \in \mathcal{B}(X)$ and $\|A\| \leq M$. Note also that, for every $n \in \mathbb{N}$,

$$|\lambda_n| = \|\lambda_n u_n\| = \|Au_n\| \leq \|A\|$$

so that

$$\sup_{n \in \mathbb{N}} |\lambda_n| \leq \|A\|.$$

Taking $M = \sup_{n\in\mathbb{N}} |\lambda_n|$, we also obtain $M \leq \|A\|$. Thus, we proved that $\|A\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

Taking $X = \ell^2$ and $u_n = e_n$, $n \in \mathbb{N}$, Example 2.1.2 becomes a special case. □

Example 2.1.10 Let X and Y be inner product spaces and $A \in \mathcal{B}(X, Y)$. Then

$$\|A\| = \sup\{|\langle Ax, y\rangle| : x \in X,\ y \in Y \text{ with } \|x\| = 1 = \|y\|\}.$$

□

2.1.3 Conditions for continuity

In the following theorem we specify a necessary and sufficient condition for a linear functional on a general normed linear space to be continuous.

Theorem 2.1.6 *Suppose X is a normed linear space and $f : X \to \mathbb{K}$ is a nonzero linear functional. Then f is continuous if and only if $N(f)$ is closed, and in that case,*

$$\|f\| = \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))}$$

for any $x_0 \notin N(f)$.

Proof. Clearly, if f is continuous, then $N(f)$ is closed.

Conversely, suppose $N(f)$ is closed. Let $x_0 \in X$ with $f(x_0) \neq 0$. Then we know that $d := \operatorname{dist}(x_0, N(f)) > 0$. Now, every $x \in X$ can be expressed as $x = y + z$, where

$$y = x - \frac{f(x)}{f(x_0)} x_0, \quad z = \frac{f(x)}{f(x_0)} x_0.$$

Note that $y \in N(f)$. Thus, for $x \in X$,

$$\operatorname{dist}(x, N(f)) = \operatorname{dist}(z, N(f)) = \left|\frac{f(x)}{f(x_0)}\right| \operatorname{dist}(x_0, N(f))$$

and hence,

$$|f(x)| \leq \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))} \operatorname{dist}(x, N(f)) \leq \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))} \|x\|.$$

Therefore, $f \in X'$ and

$$\|f\| \le \frac{|f(x_0)|}{\text{dist}\,(x_0, N(f))}.$$

Also, for every $u \in N(f)$,

$$|f(x_0)| = \|f(x_0 - u)\| \le \|f\|\|x_0 - u\|.$$

Hence, taking infimum over all $u \in N(f)$, we obtain

$$|f(x_0)| \le \|f\|\text{dist}\,(x_0, N(f)),$$

so that

$$\|f\| \ge \frac{|f(x_0)|}{\text{dist}\,(x_0, N(f))}.$$

This completes the proof. ▮

Next theorem would help in inferring the continuity of a linear operator and also in obtaining an estimate for its norm, in the case when the spaces involved are inner product spaces.

Theorem 2.1.7 *Let $A : X \to Y$ be a linear operator between inner product spaces X and Y. Then $A \in \mathcal{B}(X, Y)$ if and only if there exists $\beta > 0$ such that*

$$|\langle Ax, y\rangle| \le \beta\|x\|\,\|y\| \quad \forall\,(x, y) \in X \times Y, \qquad (*)$$

and in that case

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = 1 = \|y\|\} \le \beta.$$

Proof. Suppose $A \in \mathcal{B}(X, Y)$. Then for every $(x, y) \in X \times Y$, by Cauchy Schwarz inequality, we have

$$|\langle Ax, y\rangle| \le \|Ax\|\,\|y\| \le \|A\|\,\|x\|\,\|y\|.$$

Thus $(*)$ is satisfied with $\beta = \|A\|$ and

$$\sup\{|\langle Ax, y\rangle| : \|x\| = 1 = \|y\|\} \le \|A\|. \qquad (**)$$

Conversely, suppose there exists $\beta > 0$ such that $(*)$ holds. We know that for every $x \in X$,

$$\|Ax\| = \sup\{|\langle Ax, v\rangle| : v \in Y,\ \|v\| = 1\}.$$

Hence,

$$\|Ax\| = \sup\left\{\frac{|\langle Ax, y\rangle|}{\|y\|} : y \in Y,\ \|y\| \neq 0\right\} \leq \beta\|x\|$$

so that $A \in \mathcal{B}(X, Y)$ and $\|A\| \leq \beta$. Also, for $(x, y) \in X \times Y$ with $\|x\| = 1 = \|y\|$,

$$\|Ax\| = \sup\{|\langle Ax, y\rangle| : y \in Y,\ \|y\| = 1\},$$

so that

$$\|A\| \leq \sup\{|\langle Ax, y\rangle| : \|x\| = 1 = \|y\|\}.$$

This, together with $(**)$ shows that

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = 1 = \|y\|\}.$$

Thus the proof is over. ∎

Next theorem provides a sufficient condition for a linear operator to have a continuous inverse.

Theorem 2.1.8 *Let $A : X \to Y$ be a linear operator between normed linear spaces X and Y. Suppose there exists $\gamma > 0$ such that*

$$\|Ax\| \geq \gamma\|x\| \quad \forall\, x \in X.$$

Then

(i) *A is injective,*

(ii) *$A^{-1} : R(A) \to X$ is continuous, and*

(iii) *$\|A^{-1}\| \leq 1/\gamma$.*

Proof. It is clear that A is injective. Then, for every $y \in R(A)$, if $x \in X$ is the unique element in X such that $Ax = y$, then we obtain

$$\|y\| = \|Ax\| \geq \gamma\|x\| = \|A^{-1}y\|.$$

Thus, A^{-1} is continuous and $\|A^{-1}\| \leq 1/\gamma$. ∎

Definition 2.1.3 A linear operator $A : X \to Y$ is said to be **bounded below** if there exists $\gamma > 0$ such that

$$\|Ax\| \geq \gamma\|x\| \quad \forall\, x \in X.$$

◊

The following two corollaries are immediate from Theorem 2.1.8.

Corollary 2.1.9 *Let $A : X \to Y$ be a linear operator between nonzero inner product spaces X and Y. Suppose there exists $\gamma > 0$ such that*

$$|\langle Ax, y\rangle| \geq \gamma \|x\| \, \|y\| \quad \forall \, (x, y) \in X \times Y.$$

Then the conclusions in Theorem 2.1.8 hold.

Corollary 2.1.10 *Let $A : X \to X$ be a linear operator on an inner product space X. Suppose there exists $\gamma > 0$ such that*

$$|\langle Ax, x\rangle| \geq \gamma \|x\|^2 \quad \forall \, x \in X.$$

Then the conclusions in Theorem 2.1.8 hold.

Now, we deduce a theorem which is important in view of its applications to the theory of partial differential equations.

Theorem 2.1.11 *Let X be a Hilbert space and $A \in \mathcal{B}(X)$ be such that there exist $\gamma > 0$ satisfying*

$$|\langle Ax, x\rangle| \geq \gamma \|x\|^2 \quad \forall \, x \in X.$$

Then A is bijective, $A^{-1} \in \mathcal{B}(X)$ and $\|A^{-1}\| \leq 1/\gamma$.

Proof. By Corollary 2.1.10, A is injective, $A^{-1} : R(A) \to X$ is continuous and $\|A^{-1}\| \leq 1/\gamma$. Hence, it is enough to prove that $R(A) = X$. Now, the condition on A implies that $R(A)$ is closed and $R(A)^{\perp} = \{0\}$. Hence, by projection theorem, $R(A) = X$. ∎

2.2 Riesz Representation Theorem

Let X be an inner product space. Cooresponding to an element $u \in X$, consider $f_u : X \to \mathbb{K}$ defined by

$$f_u(x) = \langle x, u\rangle, \quad x \in X.$$

Clearly, f_u is a linear functional. Also, by Cauchy Schwarz inequality,

$$|f_u(x)| = |\langle x, u\rangle| \leq \|u\| \, \|x\|, \quad x \in X$$

so that $f \in X'$. Also, since $\|f_u(u)| = \|u\|^2$ we have $\|f_u\| = \|u\|$.

What about the converse? Is every continuous linear functional on X is of the form f_u for some $u \in X$? The answer is in negative as the following example shows.

Example 2.2.1 Let $X = c_{00}$ with ℓ^2–inner product. Consider the linear functional f on X defined by

$$f(x) = \sum_{j=1}^{\infty} \frac{x(j)}{j}, \quad x \in c_{00}.$$

Note that, by Schwarz inequality,

$$|f(x)| \le \sum_{j=1}^{\infty} \frac{|x(j)|}{j} \le \|x\|_2 \sum_{j=1}^{\infty} \frac{1}{j^2}, \quad x \in c_{00}.$$

Hene, $f \in X'$. But, there is no $u \in c_{00}$ such that $f(x) = \langle x, u\rangle$ for all $x \in c_{00}$. To see this, suppose there exists $u \in c_{00}$ such that $f(x) = \langle x, u\rangle$ for all $x \in c_{00}$. Then, in particular, we have

$$\frac{1}{k} = f(e_k) = \langle e_k, u\rangle = \overline{u(k)} \quad \forall k \in \mathbb{N}.$$

This is a contradiction to the fact that $u \in c_{00}$. $\square$

Now, we show that we do have an affirmative answer to the question raised above if X is a Hilbert space.

Theorem 2.2.1 (Riesz Representation Theorem) *Let X be a Hilbert space. Then for every $f \in X'$, there exists a unique $u_f \in X$ such that*

$$f(x) = \langle x, u_f\rangle, \quad x \in X.$$

Further, $\|u_f\| = \|f\|$.

Proof. Let $f \in X'$. Let us settle the uniqueness issue first. Suppose $u_1, u_2 \in X$ be such that

$$f(x) = \langle x, u_1\rangle \quad \text{and} \quad f(x) = \langle x, u_2\rangle$$

for all $x \in X$. Then we have

$$\langle x, u_1 - u_2\rangle = 0 \quad \forall\, x \in X,$$

so that $u_1 = u_2$.

Next, if $f = 0$, then $u = 0$ serves the purpose. So, assume that $f \neq 0$. Then, by projection theorem (Theorem 1.5.6) $N(f)^{\perp}$ is a nonzero proper closed subspace. Let $x_0 \in N(f)^\perp$ such that $\|x_0\| = 1$. Now, let $x \in X$. Since $x = y + z$ with

$$y = x - \frac{f(x)}{f(x_0)} x_0, \quad z = \frac{f(x)}{f(x_0)} x_0,$$

and since $y \in N(f)$ and $z \in N(f)^\perp$, we have

$$\langle x, x_0 \rangle = \frac{f(x)}{f(x_0)} \langle x_0, x_0 \rangle = \frac{f(x)}{f(x_0)}.$$

Thus,

$$f(x) = \langle x, u_f \rangle \quad \text{with} \quad u_f = \overline{f(x_0)} x_0.$$

The fact that $\|f\| = \|u_f\|$ follows, since $|f(x)| \le \|u_f\| \, \|x\|$ for all $x \in X$ and $|f(u_f)| = \|u_f\|^2$. ∎

The terminology defined below will be used in the due course.

Definition 2.2.1 Let X and Y be linear spaces. Then a function $T : X \to Y$ is called a **conjugate linear** if

$$T(x + y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \bar{\alpha} T(x)$$

for all $(x, y) \in X \times Y$ and $\alpha \in \mathbb{K}$. ◊

Remark 2.2.1 Let X be a Hilbert space, and for $f \in X'$, let u_f is the unique element in X obtained as in Riesz representation theorem. Then

$$\langle f, g \rangle' := \langle u_g, u_g \rangle, \quad f, g \in X'$$

defines an inner product on X' and $T : X' \to X$ defined by

$$Tf = u_f, \quad f \in X',$$

is a surjective isometry which is also conjugate linear, i.e., for every $f, g \in X'$, and $\alpha \in \mathbb{K}$,

$$T(f + g) = Tf + Tg \quad \text{and} \quad T(\alpha f) = \bar{\alpha} Tf.$$

◊

In view of the above remark, we can identify X' with X if X is a Hilbert space.

Often, certain problems in partial differential equations can be converted into the problem of finding a unique $u \in X$ such that

$$\varphi(u, v) = f(v) \quad \forall\, v \in X,$$

where X is a Hilbert space, f is a continuous linear functional on X, and the function $\varphi : X \times X \to \mathbb{K}$ is such that for each $y \in X$, $x \mapsto \varphi(x, y)$ is linear on X and for each $x \in X$, $y \mapsto \varphi(x, y)$ is conjugate linear on X. Riesz representation theorem (Theorem 2.2.1) and Theorem 2.1.11 can be effectively used in showing the existence of such solutions.

Definition 2.2.2 Let X and Y be inner product spaces. A function $\varphi : X \times Y \to \mathbb{K}$ is said to be a **sesquilinear form** on an inner product space $X \times Y$ if for each $y \in Y$,

$$x \mapsto \varphi(x, y)$$

is a linear functional on X and for each $x \in X$,

$$y \mapsto \varphi(x, y)$$

is a conjugate linear on Y. $\Diamond$

Theorem 2.2.2 *Let X be a Hilbert space, Y be an inner product space, and $\varphi : X \times Y \to \mathbb{K}$ be a sesquilinear form on $X \times Y$. Suppose there exist β such that*

$$|\varphi(x, y)| \leq \beta \|x\| \, \|y\| \quad \forall\, (x, y) \in X \times Y.$$

Then there exits a unique $B \in \mathcal{B}(Y, X)$ such that

$$\varphi(x, y) = \langle x, By \rangle \quad \forall\, (x, y) \in X \times Y$$

and in that case $\|B\| \leq \beta$.

Proof. Let $y \in X$. Since $x \mapsto \varphi(x, y)$ is a continuous linear functional on X, by Riesz representation theorem, there exists a unique $z_y \in X$ such that

$$\varphi(x, y) = \langle x, z_y \rangle \qquad \forall\, x \in X.$$

Let $By := z_y$, $y \in X$. Note that, for every $x \in X$ and $y_1, y_2 \in Y$ and $\alpha \in \mathbb{K}$,

$$\begin{aligned} \langle x, B(\alpha y_1 + y_2)\rangle &= \varphi(x, \alpha y_1 + y_2) \\ &= \bar{\alpha}\varphi(x, y_1) + \varphi(x, y_2) \\ &= \bar{\alpha}\langle x, By_1\rangle + \langle x, By_2\rangle \\ &= \langle x, \alpha By_1 + By_2\rangle. \end{aligned}$$

Hence, $B : Y \to X$ is a linear operator on X. Also, we have

$$|\langle x, By\rangle| = |\varphi(x, y)| \leq \beta\|x\|\,\|y\| \qquad \forall\, x, y \in X,$$

ao that $B \in \mathcal{B}(Y, X)$ and $\|B\| \leq \beta$. It is easy to see that such an operator B is unique. ∎

Theorem 2.2.3 (Lax-Milgram theorem) *Let X be a Hilbert space and $\varphi : X \times X \to \mathbb{K}$ be a sesquilinear form on X. Suppose there exist $\beta, \gamma > 0$ such that*

$$|\varphi(x, y)| \leq \beta\|x\|\,\|y\| \quad \forall\, x, y \in X, \tag{i}$$

$$|\varphi(x, x)| \geq \gamma\|x\|^2 \quad \forall\, x \in X. \tag{ii}$$

Then, for every $f \in X'$, there exists a unique $u \in X$ such that

$$\varphi(x, u) = f(x) \quad \forall\, x \in X,$$

and in that case $\|u\| \leq \dfrac{1}{\gamma}\|f\|$.

Proof. Let us settle the uniqueness issue first: Suppose there exist u_1, u_2 such that

$$\varphi(x, u_1) = f(x) = \varphi(x, u_2) \qquad \forall\, x \in X.$$

Then, we have

$$\varphi(x, u_1 - u_2) = 0 \quad \forall\, x \in X.$$

This implies $\varphi(u_1 - u_2, u_1 - u_2) = 0$, which implies, by condition (ii), $u_1 - u_2 = 0$.

Now, the rest of the results: By Riesz representation theorem, there exists a unique $v \in X$ be such that

$$f(x) = \langle x, v\rangle \quad \forall\, x \in X,$$

and in that case we also have $\|f\| = \|v\|$.

By Theorem 2.2.2, there exists a unique $B \in \mathcal{B}(X)$ such that

$$\varphi(x, y) = \langle x, By \rangle \quad \forall\, x, y \in X.$$

Note that

$$|\langle x, Bx \rangle| = |\varphi(x, x)| \geq \gamma \|x\|^2 \quad \forall\, x \in X.$$

Thus, $B \in \mathcal{B}(X)$ satisfies the assumption in Theorem 2.1.11. Therefore, there exists a unique $u \in X$ such that

$$Bu = v \quad \text{and} \quad \|u\| \leq \frac{1}{\gamma}\|v\|.$$

Thus,

$$\varphi(x, u) = \langle x, Bu \rangle = \langle x, v \rangle = f(x), \qquad \forall x \in X,$$

and

$$\|u\| \leq \frac{1}{\gamma}\|v\| = \frac{1}{\gamma}\|f\|.$$

This completes the proof. ∎

2.2.1 Adjoint of an operator

In Theorem 1.5.10 we have seen that if P is an orthogonal projection on an inner product space X, then

$$\langle Px, y \rangle = \langle x, Py \rangle \quad \forall\, x, y \in X.$$

Definition 2.2.3 A linear operator $A : X \to X$ on an inner product space X is called a **self adjoint operator** if

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall\, x, y \in X.$$

◊

Here are a few examples of self adjoint operator.

Example 2.2.2 Let $X = \mathbb{K}^n$ with $\|\cdot\|_2$ and $A : \mathbb{K}^n \to \mathbb{K}^n$ be the linear operator induced by an $n \times n$ matrix (a_{ij}) which satisfies

$$a_{ij} = \overline{a}_{ji} \quad \forall\, i, j = 1, \ldots, n.$$

Then we see that A is a self adjoint operator. □

Example 2.2.3 Let $X = \ell^2$, and $a_{ij} \in \mathbb{K}$ be such that

$$\beta := \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty \quad \text{and} \quad \gamma := \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty.$$

We have seen in Example 2.1.5 that

$$Ax = \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} a_{ij} x(j)\Big) e_i$$

is well defined for each $x \in \ell^2$, $Ax \in \ell^2$, $A \in \mathcal{B}(\ell^2)$ and $\|A\| \le \sqrt{\beta\gamma}$. Suppose, in addition, that

$$a_{ij} = \overline{a}_{ji} \quad \forall\, i, j \in \mathbb{N}.$$

Then, using the representation $y = \sum_{k=1}^{\infty} y(k) e_k$, we can see that

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \in \ell^2.$$

Thus, A is a self adjoint operator.

In particular, if (λ_n) is a bounded sequence of real numbers, then the operator

$$x \mapsto (\lambda_1 x(1), \lambda_2 x(2), \ldots,), \quad x \in \ell^2,$$

is a self adjoint operator. □

Example 2.2.4 Let $k(\cdot,\cdot) \in C([a,b]\times[a,b])$, and for $x \in C[a,b]$, let

$$(Ax)(s) = \int_a^b k(s,t)x(t)dt, \quad s \in [a,b].$$

Let $X = C[a,b]$ with the norm $\|\cdot\|_2$. We have seen in Example 2.1.6 that $A \in \mathcal{B}(X)$ and $\|A\| \le \int_a^b \int_a^b |k(s,t)|^2 dt$. If, in additiion,

$$k(s,t) = \overline{k(t,s)} \quad \forall s, t \in [a,b],$$

then we see that

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, x, y \in C[a,b],$$

so that, in this case, A is a self adjoint operator on X. □

There are plenty of examples of linear operators on inner product spaces which are not self adjoint. However, corresponding to a linear operator A on X, one may be able to find an operator $B : X \to X$ such that

$$\langle Ax, y\rangle = \langle x, By\rangle \quad \forall\, x, y \in X.$$

Note that if such an operator B exists, then it is unique.

Definition 2.2.4 Let X and Y be inner product spaces and let $A : X \to Y$ be a linear operator. If there is a linear operator $B : Y \to X$ such that

$$\langle Ax, y\rangle = \langle x, By\rangle \quad \forall\, x \in X,\ y \in Y,$$

then B is called the **adjoint** of A, and it is denoted by A^*. ◊

- A linear operator $A : X \to X$ on an inner product space X is self adjoint if and only if A^* exists and $A^* = A$.

A linear operator between inner product spaces need not have an adjoint as the following examples shows.

Example 2.2.5 Let $X = c_{00}$ be with ℓ^2-inner product and let $A : X \to X$ be defined by

$$Ax = \Big(\sum_{j=1}^{\infty} \frac{x(j)}{j}\Big) e_1, \quad x \in c_{00}.$$

Then for every $x, y \in c_{00}$, we have

$$\langle Ax, y\rangle = \overline{y(1)} \sum_{j=1}^{\infty} \frac{x(j)}{j}.$$

In particular,

$$\langle Ae_n, e_1\rangle = \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$$

Assume for a moment that this A has an adjoint, say B. Then we have

$$\frac{1}{n} = \langle Ae_n, e_1\rangle = \langle e_n, Be_1\rangle = \overline{(Be_1)(n)} \quad \forall\, n \in \mathbb{N}.$$

This is a contradiction to the fact that $Be_1 \in c_{00}$. Thus, we have proved that the operator A does not have an adjoint. □

However, every bounded operator between Hilbert spaces does have the adjoint, as the following theorem shows.

Theorem 2.2.4 *Let X and Y be Hilbert spaces and $A \in \mathcal{B}(X, Y)$. Then A^* exists and $A^* \in \mathcal{B}(X)$. Further,*

$$\|A^*\| = \|A\| \quad \textit{and} \quad \|A^*A\| = \|A\|^2.$$

Proof. Note that $\varphi : X \times Y \to \mathbb{K}$ defined by

$$\varphi(x, y) = \langle Ax, y\rangle, \quad (x, y) \in X \times Y,$$

is a sesquilinear functional. Hence, by Theorem 2.2.2, there exists a unique $B \in \mathcal{B}(Y, X)$ such that

$$\varphi(x, y) = \langle x, By\rangle, \quad (x, y) \in X \times Y.$$

Thus, $B = A^*$. From the relation

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \quad (x, y) \in X \times Y,$$

it follows, using Cauchy Schwarz inequality that $\|A\| = \|B\|$. Further,

$$\|A^*A\| \le \|A^*\| \, \|A\| = \|A\|^2$$

and for every $x \in X$,

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \le \|A^*Ax\| \, \|x\| \le \|A^*A\| \, \|x\|^2.$$

From this, we obtain,

$$\|A\|^2 \le \|A^*A\|.$$

Thus, we have proved $\|A^*A\| = \|A\|^2$. This completes the proof. ∎

We observe the following facts (Exercise):

1. Let X and Y be Hilbert spaces, and $A, B \in \mathcal{B}(X, Y)$ and $\alpha \in \mathbb{K}$. Then
$$(A^*)^* = A, \quad (A + \alpha B)^* = A^* + \bar{\alpha}B^*.$$

2. Let X, Y, Z be Hilbert spaces, and $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$. Show that $(BA)^* = A^*B^*$.

Remark 2.2.2 We have seen that adjoint for a linear operator need not exist if the space is an incomplete inner product space.

Can we weaken the requirement in the definition so that an adjoint always exist?

Suppose $A : X \to Y$ is a linear operator between inner product spaces. Let us consider the set

$$Y_0 := \{y \in Y : \exists\, z \in X \text{ such that } \langle Ax, y\rangle = \langle x, z\rangle\, \forall\, x \in X\}.$$

It can be easily seen that Y_0 is a subspace of Y and for every $y \in Y_0$ there exists a unique $z_y \in X$ such that

$$\langle Ax, y\rangle = \langle x, z_y\rangle \quad \forall\, x \in X.$$

Thus, we can define $B : Y_0 \to X$ by

$$By = z_y, \quad y \in Y_0,$$

and we see that $B : Y_0 \to X$ is a linear operator satisfying

$$\langle Ax, y\rangle = \langle x, By\rangle \quad \forall\, x \in X,\ y \in Y_0.$$

The above B may also be called an *adjoint* of A. The problem with this definition is that the the space Y_0 may be too *small* or the operator B may be the zero operator. For instance, in Example 2.2.5, we have $e_1 \notin Y_0$ and for $k = 2, 3, \ldots$,

$$\langle Ax, e_k\rangle = 0 \quad \forall\, x \in X,$$

so that

$$Y_0 = \text{span}\,\{e_k : k = 2, 3, \ldots\} \quad \text{and} \quad By = 0 \quad \forall\, y \in Y_0.$$

◊

2.2.2 Self adjoint, normal and unitary operators

Let X be Hilbert space and $A \in \mathcal{B}(X, Y)$. Then we know that

A is self adjoint if and only if $A^* = A$.

Definition 2.2.5 Let X be Hilbert space and $A \in \mathcal{B}(X, Y)$. Then A is said to be a

(a) **normal operator** if $A^*A = AA^*$,

(c) **unitary operator** if $A^*A = I = AA^*$. ◊

Theorem 2.2.5 *Let X be a Hilbert space. If $A \in \mathcal{B}(X)$ is a self adjoint operator, then*

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in X, \|x\| = 1\}.$$

Proof. Let $A \in \mathcal{B}(X)$ be a self adjoint operator. Clearly,

$$\gamma := \sup\{|\langle Ax, x\rangle| : x \in X, \|x\| = 1\} \leq \|A\|.$$

Next, let $x \in X$ be such that $\|x\| = 1$ and $\|Ax\| \neq 0$. It is enough to show that $\|Ax\| \leq \gamma$.

First we observe, using the self adjointness of A, that for every $y \in X$,

$$\langle A(x+y), (x+y)\rangle - \langle A(x-y), (x-y)\rangle = 4\mathrm{Re}\langle Ax, y\rangle.$$

Thus,

$$\begin{aligned}
\mathrm{Re}\langle Ax, y\rangle &= \frac{1}{4}\Big(\langle A(x+y), (x+y)\rangle - \langle A(x-y), (x-y)\rangle\Big) \\
&= \frac{1}{4}\Big(|\langle A(x+y), (x+y)\rangle| + |\langle A(x-y), (x-y)\rangle|\Big) \\
&\leq \frac{1}{4}\gamma\,(\|x+y\|^2 + \|x-y\|^2).
\end{aligned}$$

Since $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$, by using parallelogram law, we have

$$\mathrm{Re}\langle Ax, y\rangle \leq \frac{\gamma}{2}\,(\|x\|^2 + \|y\|^2).$$

Now, taking $y = \dfrac{Ax}{\|Ax\|}$, we obtain $\|Ax\| = \mathrm{Re}\langle Ax, y\rangle \leq \gamma$. ∎

The proof of the following corollary is immediate.

Corollary 2.2.6 *Let X be a Hilbert space and $A \in \mathcal{B}(X)$ be a self adjoint operator. Then*

$$A = 0 \iff \langle Ax, x\rangle = 0 \quad \forall\, x \in X.$$

The above corollary shows that a self adjoint operator A is uniquely determined by its values $\langle Ax, x\rangle$, $x \in X$. Indeed, if A_1 and A_2 are self adjoint operators on a Hilbert sapce X such that $\langle A_1x, x\rangle = \langle A_2x, x\rangle$ for all $x \in X$, then

$$\langle (A_1 - A_2)x, x\rangle = 0 \quad \forall\, x \in X$$

so that by by Corollary 2.2.6, $A_1 = A_2$.

Theorem 2.2.7 *Let X be a Hilbert space and $A \in \mathcal{B}(X, Y)$.*

(i) *A is a normal operator if and only if $\|Ax\| = \|A^*x\|$ for every $x \in X$.*

(ii) *A is a unitary operator if and only if $\|Ax\| = \|x\|$ for every $x \in X$ and A is surjective.*

Proof. Observe that for $x \in X$,

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle x, A^*Ax\rangle,$$

$$\|A^*x\|^2 = \langle A^*x, A^*x\rangle = \langle x, AA^*x\rangle.$$

Thus, for $x \in X$,

$$\|Ax\| = \|A^*x\| \iff \langle x, (A^*A - AA^*)x\rangle = 0,$$

$$\|Ax\| = \|x\| \iff \langle x, (A^*A - I)x\rangle = 0,$$

Since $A^*A - AA^*$ and $A^*A - I$ are self adjoint, by Corollary 2.2.6,

(i) $A^*A - AA^* = 0$ if and only if $\|Ax\| = \|A^*x\|$ for every $x \in X$,

(ii) $A^*A = I$ if and only if $\|Ax\| = \|x\|$ for every $x \in X$.

Note that, if $A^*A = I$, then A is injective, and if, in addition, A is surjective, then A is bijective and $A^* = A^{-1}$ so that A is unitary. This completes the proof. ∎

2.3 The Dual Space of Certain Spaces

We know that if X is a Hilbert space, then its dual can be identified with X by a conjugate linear isometry. Also, we know that for every normed linear space X, its dual space X' is a Banach space with respect to the norm

$$\|f\| := \sup\{|f(x)| : \|x\| \leq 1\}, \quad f \in X'.$$

So, in general, we cannot expect that X is linearly isometric with X', not even necessary to be homeomorphic with X'. Of course, if X is finite dimensional, then X' is of the same dimension as that of X, and X' is linearly homeomorphic with X.

In the following we give some representations of dual spaces of certain sequence spaces.

2.3.1 Dual of some sequence spaces

First, let us consider the sequence space c_{00} with norms $\|\cdot\|_p$ for $1 \le p \le \infty$.

Theorem 2.3.1 *Let $X_p = c_{00}$ with $\|\cdot\|_p$ for $1 \le p \le \infty$ and let q be the conjugate exponent of p. Then for every $f \in X_p'$, there exists a unique $u_f \in \ell^q$ such that*

$$f(x) = \sum_{i=1}^{\infty} x(i)\overline{u_f(i)} \quad \forall\, x \in X_p,$$

and the map $f \mapsto u_f$ is a surjective isometry. In particular, X_p' is linearly isometric with ℓ^q,

Proof. Let $f \in X_p'$, $u := (f(e_1), f(e_2), \ldots)$ and $x \in X_p$. Since $\{e_1, e_2, \ldots\}$ is a basis of X_p, we have

$$f(x) = \sum_{i=1}^{\infty} x(i) f(e_i) = \sum_{i=1}^{\infty} x(i) u(i).$$

First we show that $u \in \ell^q$ and $\|u\|_q \le \|f\|$.

Note that

$$|u(i)| = |f(e_i)| \le \|f\| \quad \forall\, i \in \mathbb{N}.$$

Hence, $u \in \ell^\infty$ and $\|u\|_\infty \le \|f\|$. Thus, for $p = 1$, we have $u \in \ell^q$ and $\|u\|_q \le \|f\|$, where $q = \infty$. Next, let $1 < p \le \infty$ and for $n \in \mathbb{N}$, let

$$x_n(i) = \begin{cases} |u(i)|^q / u(i), & u(i) \ne 0,\ i \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have $x_n \in c_{00}$ and

$$\sum_{i=1}^{n} |u(i)|^q = |f(x_n)| \le \|f\|\, \|x_n\|_p.$$

If $p = \infty$, then $q = 1$ and $\|x_n\|_\infty \le 1$ so that in this case, $u \in \ell^1$ and $\|u\|_1 \le \|f\|$. So, let $1 < p < \infty$. Then, using the fact that $pq - p = q$, we have

$$\|x_n\|_p^p = \sum_{i=1}^{n} |x_n(i)|^p = \sum_{i=1}^{n} |u(i)|^q.$$

Therefore,

$$\sum_{i=1}^{n} |u(i)|^q = |f(x_n)| \le \|f\| \, \|x_n\|_p = \|f\| \left(\sum_{i=1}^{n} |u(i)|^q \right)^{1/p}$$

so that

$$\left(\sum_{i=1}^{n} |u(i)|^q \right)^{1/q} \le \|f\|; \quad \text{equivalently,} \quad \sum_{i=1}^{n} |u(i)|^q \le \|f\|^q.$$

Hence, $u \in \ell^q$ and $\|u\|_q \le \|f\|$. By Hölder's inequality, we also have

$$|f(x)| \le \|x\|_p \|u\|_q.$$

Thus, we have proved that $u \in \ell^q$ and $\|u\|_q = \|f\|$.

For $f \in X'_p$, let the element $u \in \ell^q$ defined above be denoted by u_f. We have already shown that the function $T : X'_p \to \ell^q$ defined by $T(f) = u_f$ is an isometry. It can be easily seen that it is linear as well. Now, we show that T is onto. For this, let $y \in \ell^q$, and let $f : X_p \to \mathbb{K}$ be defined by

$$f(x) = \sum_{i=1}^{\infty} x(i)y(i), \quad x \in X_p.$$

Then, by Hölder's inequality, $f \in X'_p$ and $\|f\| \le \|y\|_q$, and $f(e_i) = y(i)$ for all $i \in \mathbb{N}$. Hence, $y = u_f$, i.e., $T(f) = y$. ▮

Next, we show that the dual of ℓ^p can be identified with ℓ^q for the case $1 \le p < \infty$. For this, first we observe the following lemma.

Lemma 2.3.2 *Let X be a normed linear space and X_0 be a dense subspace of X. If $f_0 : X_0 \to \mathbb{K}$ is a continuous linear functional, then there exists a unique continuous linear functional $f : X \to \mathbb{K}$ such that*

$$f(x) = f_0(x) \quad \forall\, x \in X_0 \quad \textit{and} \quad \|f\| = \|f_0\|.$$

Proof. Let $f_0 : X_0 \to \mathbb{K}$ be a continuous linear functional. For $x \in X$, let (x_n) in X_0 be such that $\|x_n - x\| \to 0$ as $n \to \infty$. Then, for every $m, n \in \mathbb{N}$, we have

$$|f_0(x_n) - f_0(x_m)| \le \|f_0\| \, \|x_n - x_m\|.$$

Hence $(f_0(x_n))$ is a Cauchy sequence in $\mathbb{K}$. Define

$$f(x) := \lim_{n\to\infty} f_0(x_n), \quad x \in X.$$

Then, it follows that $f : X \to \mathbb{K}$ is linear and $f(x) = f_0(x)$ for all $x \in X_0$. Further,

$$|f(x)| = \lim_{n\to\infty} |f_0(x_n)| \le \|f_0\| \lim_{n\to\infty} \|x_n\| = \|f_0\|\,\|x\|.$$

Hence, $f \in X'$ and $\|f\| \le \|f_0\|$. Clearly, $\|f_0\| \le \|f\|$. Thus, existence result is proved. For the uniqueness, suppose, there exists $\tilde{f} \in X'$ such that

$$\tilde{f}(x) = f_0(x) \quad \forall\, x \in X_0 \quad \text{and} \quad \|\tilde{f}\| = \|f_0\|.$$

Then, for $x \in X$, taking (x_n) in X_0 such that $\|x_n - x\| \to 0$ as $n \to \infty$, we have

$$\tilde{f}(x) = \lim_{n\to\infty} \tilde{f}(x_n) = \lim_{n\to\infty} f_0(x_n) = f(x).$$

Thus, $\tilde{f} = f$. ∎

Theorem 2.3.3 *Let $1 \le p < \infty$. Then, for every $f \in (\ell^p)'$, there exists a unique $u_f \in \ell^q$ such that*

$$f(x) = \sum_{i=1}^{\infty} x(i) u_f(i) \quad \forall\, x \in \ell^p,$$

and the map $f \mapsto u_f$ is a surjective linear isometry from $(\ell^p)'$ to ℓ^q.

Proof. For $1 \le p < \infty$, let $X_p = c_{00}$ with the norm $\|\cdot\|_p$. Let $f \in (\ell^p)'$ and $g : X_p \to \mathbb{K}$ be defined by

$$g(x) = f(x) \quad \forall\, x \in X_p.$$

Note that $g \in X_p'$. By Theorem 2.3.1, there exists a unique $u \in \ell^q$ such that

$$g(x) = \sum_{i=1}^{\infty} x(i) u(i) \quad \forall\, x \in X_p,$$

and $\|g\| = \|u\|_q$. Since c_{00} is dense in ℓ^p for $1 \le p < \infty$ (see Example 1.3.2), by Lemma 2.3.2, f is the unique extension of g such that

$\|f\| = \|g\|$ so that we also have $\|f\| = \|u\|$. Further, for $x \in \ell^p$, let (x_n) be in c_{00} such that $\|x - x_n\|_p \to 0$ as $n \to \infty$. Then, we have

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sum_{i=1}^{n} x(i)u(i) = \sum_{i=1}^{\infty} x(i)u(i).$$

Clearly, $f(e_i) = u(i)$ for all $i \in \mathbb{N}$. From this, uniqueness of u also follows. To see the surjectivity of the map $f \mapsto u_f := u$, let $y \in \ell^q$. Define $f : \ell^p \to \mathbb{K}$ by

$$f(x) = \sum_{i=1}^{\infty} x(i)y(i) \quad \forall\, x \in \ell^p.$$

Then, by Hölder's inequality, $f \in (\ell^p)'$. By the argument as in the beginning of this proof, we see that $\|f\| = \|y\|_q$ and $u_f = y$. This completes the proof. ∎

Theorem 2.3.4 *Let $X = c_0$ with $\|\cdot\|_\infty$. Then, for every $f \in X'$, there exists a unique $u_f \in \ell^1$ such that*

$$f(x) = \sum_{i=1}^{\infty} x(i)u_f(i) \quad \forall\, x \in c_0,$$

and the map $f \mapsto u_f$ is a surjective linear isometry from X' to ℓ^1.

Proof. We observe that c_{00} is dense in c_0 with respect to the norm $\|\cdot\|_\infty$. Hence, the proof follows using the arguments as in the proof of Theorem 2.3.3 by replacing ℓ^p by X and ℓ^q by ℓ^1. ∎

Remark 2.3.1 It can also be shown that the dual of c (the space of convergent scalar sequences with respect to the norm $\|\cdot\|_\infty$) is linearly isometric with ℓ^1 (cf. Nair [5]). ♢

2.3.2 Dual of some function spaces

Now, we consider dual of the space $C[a,b]$ with $\|\cdot\|_\infty$ and of the space $L^p[a,b]$ for $1 \le p < \infty$. We shall state the main theorems without proofs. Interested readers may see the proofs in Nair [5]. However, we provide here all necessary details required for their statements.

Recall from Remark 1.3.2 that $L^p[a,b]$ for $1 \le p < \infty$ is the linear space of all measurable functions $x : [a,b] \to \mathbb{K}$ such that $\int_a^b |x(t)|^p dm(t) < \infty$, where $m(\cdot)$ is the Lebesgue measure on $[a,b]$.

The set $L^\infty[a,b]$ is the set of all *essentially bounded functions* on $[a,b]$, that is, $x : [a,b] \to \mathbb{K}$ belongs to $L^\infty[a,b]$ if and only if it is measurable and there exists $M > 0$ such that $|x(t)| \le M$ for almost all (a.a) $t \in [a,b]$. In fact, we do not distinguish functions in $L^p[a,b]$ which are equal almost every where on $[a,b]$. Thus, for functions $x, y \in L^p[a,b]$, we write

$$x = y \iff x(t) = y(t) \quad \text{a.a. } t \in [a,b].$$

For $x \in L^p[a,b]$ with $1 \le p \le \infty$, let

$$\|x\|_p := \begin{cases} \left(\int_a^b |x|^p d\mu\right)^{1/p}, & 1 \le p < \infty, \\ \inf\{M > 0 : |x(t)| \le M \text{ a.a. } t \in [a,b]\}, & p = \infty. \end{cases}$$

It is known (cf. Nair [5] or Rudin [10]) that

- $L^p[a,b]$ is a linear space and
- the map $x \mapsto \|x\|_p$ is a complete norm on $L^p[a,b]$.

Definition 2.3.1 A function $v : [a,b] \to \mathbb{K}$ is said to be a **function of bounded variation** on $[a,b]$ if there exists $M > 0$ such that for every partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$, we have

$$\sum_{i=1}^n |v(t_i) - v(t_{i-1})| \le M.$$

♢

- The set $BV[a,b]$ of all functions of bounded variation on $[a,b]$ is a linear space,
- The function $v \mapsto \|v\| := |v(a)| + \sup \sum_{i=1}^n |v(t_i) - v(t_{i-1})|$ is a norm on $BV[a,b]$, where the supremum is taken over all partitions of $[a,b]$, and
- $BV[a,b]$ is a Banach space with respect to the above norm.

Further, it is known (See Royden [8]) that every real valued function of bounded variation is a difference of two monotonically increasing functions. Thus, we can define Riemann-Stieltjes integral of a continuous function with respect to a function in $BV[a,b]$ in a natural way.

Definition 2.3.2 A function $v \in BV[a,b]$ is said to be a **normalized function of bounded variation** if $v(a) = 0$ and if it is right continuous at every point in $[a,b]$, i.e., for every $t \in [a,b]$, $\lim\limits_{\delta\to 0} v(t+\delta)$ exists and it is equal to $v(t)$. $\Diamond$

- The set $NBV[a,b]$ of all normalized functions of bounded variation on $[a,b]$ is a closed subspace of $BV[a,b]$.

Thus, $NBV[a,b]$ is a Banach space with respect to the norm

$$v \mapsto \|v\| := \sup \sum_{i=1}^{n} |v(t_i) - v(t_{i-1})|.$$

Now, we can state the main theorems of this subsection.

Theorem 2.3.5 *For each $y \in NBV[a,b]$, let*

$$f_y(x) := \int_a^b x(t) dy(t), \quad x \in C[a,b].$$

Then f_y is a continuous linear functional on $C[a,b]$ (with respect to $\|\cdot\|_\infty$)) and $y \mapsto f_y$ is a surjective linear isometry from $NBV[a,b]$ onto the dual of $C[a,b]$.

Theorem 2.3.6 *Let $1 \le p < \infty$ and $q > 0$ be the conugate exponent of p. For each $y \in L^q[a,b]$, let*

$$f_y(x) := \int_a^b x(t)y(t)\, dm(t), \quad x \in L^p[a,b].$$

Then f_y is a continuous linear functional on $L^p[a,b]$ and the the map $y \mapsto f_y$ is a surjective linear isometry from $L^q[a,b]$ onto the dual of $L^p[a,b]$.

2.4 Compact Operators

Definition 2.4.1 Let $A : X \to Y$ be a linear operator between normed linear spaces X and Y. We say that A is a **finite rank operator** if

$$\dim R(A) < \infty.$$

A linear operator $A : X \to Y$ is said to be of **infinite rank** if it is not of finite rank. $\Diamond$

If $A : X \to Y$ is of finite rank, then we write

$$\text{rank}\,(A) = \dim R(A).$$

Finite rank operators appear naturally in applications in the form of approximation of operators of infinite rank.

Let us illustrate the approximation procedure by one example.

Example 2.4.1 Let X and Y be Hilbert spaces, (u_n) and (v_n) be orthonormal sets in X and Y, respectively, and let (μ_n) be a bounded sequence of scalars. Define $A : X \to Y$ by

$$Ax = \sum_{j=1}^{\infty} \mu_j \langle x, u_j \rangle v_j, \quad x \in X.$$

We have seen in Example 2.1.9 that $A \in \mathcal{B}(X)$ and $\|A\| = \sup_{j\in\mathbb{N}} |\mu_j|$. Now, for each $n \in \mathbb{N}$, let $A_n : X \to Y$ be defined by

$$A_n x = \sum_{j=1}^{n} \mu_j \langle x, u_j \rangle v_j, \quad x \in X.$$

Then we have

$$\|(A - A_n)x\|^2 = \sum_{j=n+1}^{\infty} |\mu_j|^2 |\langle x, u_n \rangle|^2 \le \max_{j>n} |\mu_j|^2 \|x\|^2 \quad \forall\, x \in X.$$

Hence,

$$\|A - A_n\| \le \max_{j>n} |\mu_j|$$

so that if $\mu_n \to 0$ as $n \to \infty$, we obtain $\|A - A_n\| \to 0$ as $n \to \infty$. Note that A is an infinite rank operator, whereas $\text{rank}\,(A_n) \le n$ for every $n \in \mathbb{N}$. □

Remark 2.4.1 Example 2.4.1 shows that the limit of a sequence of finite rank operators in $\mathcal{B}(X, Y)$ need not be of finite rank. ◊

One of the important property of a finite rank operator is that image of the closed unit ball is *relatively compact.* This property is shared by a large class of operators. Recall from real analysis that a subset of a metric space is said to be **relatively compat** if its closure is compact.

Definition 2.4.2 Let X and Y be normed linear spaces. Then a linear operator $A : X \to Y$ is said to be a **compact operator** if $\{Ax : \|x\| \leq 1\}$ is relatively compact. $\Diamond$

Notation 2.4.1 We denote the set of all compact operators from X to Y by $\mathcal{K}(X, Y)$, and also we denote $\mathcal{K}(X, X)$ by $\mathcal{K}(X)$ $\Diamond$

Clearly,

$$\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y).$$

Theorem 2.4.1 *The following hold.*

(ii) *Every bounded finite rank operator is compact.*

(iii) *The identity operator on a normed linear space is compact if and only if the space is finite dimensional.*

Proof. Let X and Y be normed linear spaces.

(i) Let $A : X \to Y$ be a bounded operator of finite rank. Then $\mathrm{cl}\,\{Ax : \|x\| \leq 1\}$ is a closed and bounded subset of the finite dimensional space $Y_0 := R(A)$, so that $\mathrm{cl}\,\{Ax : \|x\| \leq 1\}$ is compact in Y_0, and hence compact in Y as well.

(ii) This follows from the fact that the closed unit ball $\{x \in X : \|x\| \leq 1\}$ is compact if and only if the space X is finite dimensional (cf. Theorem 1.3.7). ∎

The following proposition is a consequence of the fact that a subset S of a metric space Ω is compact if and only if every sequence in S has a subsequence which converges in S.

Proposition 2.4.2 *Let X and Y be normed linear spaces. A linear operator $A : X \to Y$ is compact if and only if for every bounded sequence (x_n) in X, the sequence (Ax_n) has a convergent subsequence.*

Theorem 2.4.3 *Let X and Y be normed linear spaces.*

(i) *$\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.*

(i) *If Y is a Banach space, then $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$.*

Proof. (i) Let A and B be in $\mathcal{K}(X, Y)$ and $\alpha \in \mathbb{K}$. Let (x_n) be a bounded sequence in X. In view of Proposition 2.4.2, it is enough to show that the sequence $((A + \alpha B)x_n)$ has a convergent subsequence. Since A and B are compact, by Proposition 2.4.2, there exists a subsequence (x'_n) for (x_n) and a subsequence (x''_n) for (x'_n) such that (Ax'_n) and (Bx''_n) converge, say to y and z respectively. Hence,

$$Ax''_n + \alpha Bx''_n \to z + \alpha z \quad \text{as} \quad n \to \infty.$$

(ii) Suppose Y be a Banach space. Let (A_n) be a sequence in $\mathcal{K}(X,Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$ for some $A \in \mathcal{B}(X,Y)$. We have to show that $A \in \mathcal{K}(X,Y)$. Again, let (x_n) be a bounded sequence in X, say $\|x_n\| \le c$ for all $n \in \mathbb{N}$. In view of Proposition 2.4.2, it is enough to show that the sequence (Ax_n) has a convergent subsequence. Since Y is complete, it enough to show that (Ax_n) has a Cauchy subsequence.

Since each A_k is compact, there exists a subsequence $(x_n^{(k)})$ for (x_n) such that $(A_k x_n^{(k)})$ converges. Without loss of generality, we may assume that $(x_n^{(k+1)})$ is a subsequence of $(x_n^{(k)})$ for each $k \in \mathbb{N}$. Note that, for each $k \in \mathbb{N}$, $(x_{k+n}^{(k+n)})$ is a subsequence of $(x_{k+n}^{(k)})$. Hence, $(A_k x_n^{(n)})$ converges for each $k \in \mathbb{N}$. Now, let $\varepsilon > 0$ and let $k \in \mathbb{N}$ be such that $\|A - A_k\| < \varepsilon$. Corresponding to this k, let $N \in \mathbb{N}$ be such that

$$\|A_k x_n^{(n)} - A_k x_m^{(m)}\| < \varepsilon \quad \forall n, m \ge N.$$

Then, for all $n, m \ge N$, we have

$$\begin{aligned} \|Ax_n^{(n)} - Ax_m^{(m)}\| &\le \|(A - A_k)x_n^{(n)}\| + \|(A_k x_n^{(n)} - A_k x_m^{(m)}\| \\ &\quad + \|(A_k - A)x_n^{(n)}\| \\ &\le c\varepsilon + \varepsilon + c\varepsilon \\ &= (2c+1)\varepsilon. \end{aligned}$$

Thus, $(Ax_n^{(n)})$ is a Cauchy subsequence of (Ax_n). ∎

Remark 2.4.2 We shall see in Chapter 4 that if X and Y are Hilbert spaces, then every operator in $\mathcal{K}(X,Y)$ can be approximated by a sequence of finite rank operators in $\mathcal{B}(X,Y)$. ◊

2.4.1 Examples of compact operators

Example 2.4.2 By Theorem 2.4.3 the operator A in Example 2.4.1 is a compact operator. □

Example 2.4.3 Let (λ_n) be a sequence of scalars which converges to 0, and $A : \ell^p \to \ell^p$ be defined by

$$(Ax)(i) = \lambda_i x(i), \quad i \in \mathbb{N}.$$

For $n \in \mathbb{N}$, let $A_n : \ell^p \to \ell^p$ be defined by

$$(A_n x)(i) = \begin{cases} \lambda_i x(i), & i \leq n, \\ 0, & i > n. \end{cases}$$

Then we see that

$$\|(A - A_n)x\|_p \leq \sup_{j>n} |\lambda_j| \|x\|_p \quad \forall\, x \in \ell^p$$

so that

$$\|A - A_n\| \leq \sup_{j>n} |\lambda_j| \to 0 \quad \text{as} \quad n \to \infty.$$

Note that each A_n is a finite rank bounded operator so that A_n is a compact operator, and hence by Theorem 2.4.3, A is a compact operator.

Note that, for $p = 2$, this example is a particular case of Example 2.4.2. □

For the next few examples we shall make use of *Arzela-Ascoli theorem.*

Theorem 2.4.4 (Arzela-Ascoli theorem) *A subset $\mathcal{S}$ of $C[a, b]$ is relatively compact if and only if S is pointwise bounded and equi-continuous.*

In stating the above theorem we used the following definitions: Let $\mathcal{S}$ be a set of $\mathbb{K}$-valued functions defined on metric space Ω.

1. $\mathcal{S}$ is *pointwise bounded* if for each $t \in \Omega$, there exists $M_t > 0$ such that
$$|f(t)| \leq M_t \quad \forall\, f \in S.$$

2. $\mathcal{S}$ is *equi-continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$s, t \in \Omega,\ |s - t| < \delta \implies |f(s) - f(t)| < \varepsilon \quad \forall\, f \in \mathcal{S}.$$

Example 2.4.4 (i) For $x \in C[a,b]$, define

$$(Ax)(s) = \int_a^s x(t)\,dt.$$

We have already seen that $A : C[a,b] \to C[a,b]$ is a bounded linear operator with respect to the norm $\|\cdot\|_\infty$. Further, since

$$|(Ax)(s) - (Ax)(\tau)| \le \int_\tau^s |x(t)|\,dt \le \|x\|_\infty |s-\tau|$$

for every $x \in C[a,b]$ and for every $s, \tau \in [a,b]$, it follows that the set

$$S := \{Ax : \|x\|_\infty \le 1\}$$

is bounded and equi-continuous in $C[a,b]$. Hence, by Arzela-Ascoli's theorem, S is relatively compact. Hence A is a compact operator.

(ii) Let $X = L^2[a,b]$ and

$$(Ax)(s) = \int_a^s x(t)\,dt, \quad x \in L^2[a,b].$$

Note that, for $s, \tau \in [a,b]$ with s, τ, and $x \in L^2[a,b]$, we have, $Ax \in C[a,b]$ and

$$|(Ax)(s) - (Ax)(\tau)| \le \int_s^\tau |x(t)|\,dt \le (\tau - s)^{1/2}\|x\|_2.$$

Hence, it follows that

$$S := \{Ax : \|x\|_2 \le 1\}$$

is bounded and equi-continuous in $C[a,b]$, and hence, by Arzela-Ascoli's theorem, it is relatively compact in $C[a,b]$ with respect to $\|\cdot\|_\infty$. Therefore, using the fact that

$$\|y\|_2 \le \sqrt{b-a}\,\|y\|_\infty \quad \forall\, y \in C[a,b],$$

S is relatively compact in $L^2[a,b]$. Thus, $A : L^2[a,b] \to L^2[a,b]$ is a compact operator. □

Example 2.4.5 Let $k(\cdot,\cdot)$ be a continuous function defined on $[a,b] \times [c,d]$. For $x \in L^1[a,b]$, let

$$(Ax)(s) = \int_a^b k(s,t)x(t)\,d\mu(t), \quad s \in [c,d].$$

It can seen easily that $Ax \in C[c,d]$ for all $x \in L^1[a,b]$. We show that $A: L^1[a,b] \to C[c,d]$ is a compact operator with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $L^1[a,b]$ and $C[c,d]$ respectively.

Observe that for $x \in L^1[a,b]$ and $s, \tau \in [c,d]$,

$$(Ax)(s) - (Ax)(\tau) = \int_a^b [k(s,t) - k(\tau,t)]x(t)\, d\mu(t)$$

so that

$$|(Ax)(s) - (Ax)(\tau)| \leq \Big(\sup_{t\in[a,b]} |k(s,t) - k(\tau,t)|\Big)\|x\|_1.$$

From this, it follows that $Ax \in C[c,d]$ for every $x \in C[c,d]$ and

$$\{Ax : x \in L^1[a,b],\ \|x\|_1 \leq 1\}$$

is bounded and equi-continuous in $C[c,d]$. Therefore, the operator $A: L^1[a,b] \to C[c,d]$ is compact with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $L^1[a,b]$ and $C[c,d]$ respectively. □

2.4.2 Examples of non-compact operators

Example 2.4.6 (i) Consider the right-shift operator

$$A : (\alpha_1, \alpha_2, \ldots) \mapsto (0, \alpha_1, \alpha_2, \ldots)$$

from ℓ^p to ℓ^r, where $p, r \in [1, \infty]$.

Note that the sequence (e_n), where $e_n = (\delta_{1n}, \delta_{2n}, \ldots)$, is bounded in ℓ^p, but its image (Ae_n) does not have a convergent subsequence. Indeed, for $n \neq m$,

$$\|Ae_n - Ae_m\|_r = \|e_{n+1} - e_{m+1}\|_r = \begin{cases} 1, & r = \infty \\ 2^{1/r}, & r \neq \infty. \end{cases}$$

Thus, A is not a compact operator.

(ii) Following the arguments as in (i) above, it can be seen that the left-shift operator

$$A : (\alpha_1, \alpha_2, \ldots) \mapsto (\alpha_2, \alpha_3, \ldots)$$

from ℓ^p to ℓ^r, where $p, r \in [1, \infty]$, is not a compact operator. □

Example 2.4.7 Let (λ_n) be a sequence of scalars which converges to a nonzero scalar λ, and $A : \ell^p \to \ell^p$ be defined as in Example 2.4.3, i.e.,

$$(Ax)(i) = \lambda_i x(i), \quad i \in \mathbb{N}.$$

Note that, $Ae_n = \lambda_n e_n$ for all $n \in \mathbb{N}$ so that for $n \neq m$,

$$\begin{aligned}
\|Ae_n - Ae_m\|_p &= \|\lambda_n e_n - \lambda_m e_m\|_p \\
&\geq \|\lambda_n (e_n - e_m)\|_p - \|(\lambda_n - \lambda_m) e_m\|_p \\
&= c_p |\lambda_n| - |\lambda_n - \lambda_m|,
\end{aligned}$$

where

$$c_p := \begin{cases} 1, & p = \infty \\ 2^{1/p}, & p \neq \infty. \end{cases}$$

Since $\lambda_n \to \lambda \neq 0$, there exists $N \in \mathbb{N}$ such that

$$|\lambda_n| \geq |\lambda|/2 \quad \text{and} \quad |\lambda_n - \lambda_m| < c_p|\lambda|/4 \quad \forall\, n, m \geq N.$$

Then we have

$$\|Ae_n - Ae_m\|_p \geq c_p |\lambda|/4 \quad \forall\, n, m \geq N$$

so that (Ae_n) does not have a convergent subsequence. Consequently, A is not a compact operator. □

2.5 Problems

1. Let X, Y be normed linear spaces and $A : X \to Y$ be a linear operator. Then show that the following are equivalent:
   (a) A is continuous
   (b) For every bounded subset S of X, the set $A(S)$ is bounded in Y.
   (c) The set $\{\|Ax\| : \|x\| < 1\}$ is bounded.

2. Prove that for $A \in \mathcal{B}(X, Y)$, the quantities

$$\alpha_A := \sup\{\|Ax\| : \|x\| \leq 1\},$$

$$\beta_A := \sup\{\|Ax\| : \|x\| = 1\},$$

$$\gamma_A := \sup\left\{\frac{\|Ax\|}{\|x\|} : x \neq 0\right\}$$

   are finite and are equal to $\|A\|$.

3. If $T : X \to Y$ is a linear operator such that there exists $c > 0$ and $x_0 \neq 0$ in X satisfying $\|Tx\| \leq c\|x\|$ for all $x \in X$ and $\|Tx_0\| = c\|x_0\|$, then show that $T \in \mathcal{B}(X, Y)$ and $\|T\| = c$.

4. Let X be an inner product space and $u \in X$. Prove that, for every $u \in X$, $f_u : X \to \mathbb{K}$ defined by $f_u(x) = \langle x, u \rangle$, $x \in X$, belongs to X' and $\|f_u\| = \|u\|$.

5. Let $X_p = c_{00}$ be with p-norm for $1 \leq p \leq \infty$ and $A : X \to X$ be defined by
$$(Ax)(j) = jx(j), \quad x \in c_{00}.$$
Show that A is an unbounded linear operator.

6. For $1 \leq p < \infty$, , let $X = \{x \in \ell^p : \sum_{j=1}^{\infty} j^p |x(j)|^p < \infty\}$ with $\| \cdot \|_p$ and $A : X \to \ell^p$ be defined by
$$(Ax)(j) = jx(j), \quad x \in X.$$
Show that A is an unbounded linear operator.

7. Let $k(\cdot, \cdot) \in C([a, b] \times [a, b])$. For $x \in C[a, b]$, let
$$(Ax)(s) = \int_a^b k(s, t)x(t)dt, \quad s \in [a, b].$$
For $1 \leq p \leq \infty$, if $X_p := C[a, b]$ with $\| \cdot \|_p$, then prove that $A \in \mathcal{B}(X_p, X_r)$ for any $p, r \in [1, \infty]$. Also, find an estimate for $\|A\|$ for each $p, r \in [1, \infty]$.

8. Let $X = \mathbb{K}^n$ and $Y = \mathbb{K}^m$ be with $\| \cdot \|_1$ and let (a_{ij}) be an $m \times n$ matrix over $\mathbb{K}$. For $x \in \mathbb{K}^n$, let Ax be defined by
$$(Ax)(i) = \sum_{j=1}^{n} a_{ij}x(j), \quad i = 1, \ldots, m.$$
Show that $\|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$.

9. Let $X = \mathbb{K}^n$ and $Y = \mathbb{K}^m$ be with $\| \cdot \|_\infty$ and let (a_{ij}) be an $m \times n$ matrix over $\mathbb{K}$. For $x \in \mathbb{K}^n$, let Ax be defined by
$$(Ax)(i) = \sum_{j=1}^{n} a_{ij}x(j) \quad i = 1, \ldots, m.$$
Show that $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |a_{ij}|$.

10. Let $X = \ell^1$ and let (a_{ij}) be an infinite matrix of scalars such that $\alpha_0 := \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty$. For $x \in \ell^1$, let Ax be defined by
$$(Ax)(i) = \sum_{j=1}^{\infty} a_{ij}x(j), \quad i \in \mathbb{N}.$$
Show that $A \in \mathcal{B}(\ell^1)$ and $\|A\| = \alpha_0$.

11. Let $X = \ell^\infty$ and let (a_{ij}) be an infinite matrix of scalars such that $\beta_0 := \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty$. For $x \in \ell^\infty$, let Ax be defined by
$$(Ax)(i) = \sum_{j=1}^{\infty} a_{ij}x(j), \quad i \in \mathbb{N}.$$
Show that $A \in \mathcal{B}(\ell^\infty)$ and $\|A\| = \beta_0$.

12. Let (λ_n) be a bounded sequence of scalars, and for $1 \le p \le \infty$, let
$$Ax = \sum_{n=1}^{\infty} \lambda_n x(n) e_n, \quad x \in \ell^p.$$
Show that $A \in \mathcal{B}(\ell^p)$ and $\|A\| = \sup |\lambda_n|$.

13. Show that for every $f \in (\ell^2)'$, there exists a unique $y \in \ell^2$ such that $f(x) = \sum_{j=1}^{\infty} x(j)y(j)$ for all $x \in \ell^2$.

14. Let X and Y be inner product spaces, and $A \in \mathcal{B}(X, Y)$. Prove that

    (a) $\|x\| = \sup\{|\langle x, u\rangle| : u \in X, \|u\| = 1\}$,

    (b) $\|A\| = \sup\{|\langle Ax, y\rangle : x \in X, y \in Y, \|x\| = 1 = \|y\|\}$.

15. Let $C[a, b]$ with $\|\cdot\|_\infty$. Prove that the inclusion operators

    (a) from $C[a, b] \to L^p[a, b]$ for any $p \in [1, \infty]$,

    (b) from $L^p[a, b] \to L^r[a, b]$ for any $p, r \in [1, \infty]$ with $p \ge r$

    are bounded operators.

16. Let X be a Hilbert space and $A \in \mathcal{B}(X)$ be such that there exist $\gamma > 0$ satisfying
$$|\langle Ax, x\rangle| \ge \gamma \|x\|^2 \quad \forall\, x \in X.$$
Prove that $R(A)$ is closed and $R(A)^\perp = \{0\}$.

17. Let X be a Hilbert space and for $f \in X'$, let $u_f \in X$ be the unique element obtained as in Riesz representation theorem. For f, g in X', let $\langle f, g\rangle' = \langle u_g, u_f\rangle$. Prove the following.

    (a) $\langle \cdot, \cdot\rangle'$ is an inner product on X',

    (b) X' is a Hilbert space with respect to the inner product $\langle \cdot, \cdot\rangle'$.

18. Prove the following

    (a) Let X and Y be Hilbert spaces, and $A, B \in \mathcal{B}(X, Y)$ and $\alpha \in \mathbb{K}$. Then

    $$(A^*)^* = A, \quad (A + \alpha B)^* = A^* + \bar{\alpha} B^*.$$

    (b) Let X, Y, Z be Hilbert spaces, and $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$. Show that $(BA)^* = A^* B^*$.

19. Let X_0 be a dense subspace of a normed linear space X. Prove that X_0' and X' are linearly isometric.

20. Prove Proposition 2.4.2.

21. Let A be as in Example 2.4.1. Prove that, if A is a compact operator, then 0 is the only limit point of $\{\mu_n : n \in \mathbb{N}\}$.

22. Let $k(\cdot, \cdot) \in C([a, b] \times [a, b])$ and let

    $$(Ax)(s) = \int_a^b k(s, t) x(t) dt, \quad x \in L^1[a, b].$$

    Prove that A as an operator

    (a) from $L^p[a, b] \to C[a, b]$ for any $p \in [1, \infty]$,

    (b) from $C[a, b] \to L^p[a, b]$ for any $p \in [1, \infty]$,

    (c) from $L^p[a, b] \to L^r[a, b]$ for any $p, r \in [1, \infty]$ with $p \geq r$,

    is a compact bounded operator.

    (*Hint:* Use the fact that $A : L^1[a, b] \to C[a, b]$ is a compact operator and Problem 15.)

23. Prove that a projection operator on a Banach space is compact if and only it is finite rank.