3

Some Important Theorems

3.1 Closed Graph Theorem

We know that if $A : x \to Y$ is a continuous linear operator between normed linear spaces X and Y, then for every sequence (x_n) in X,

$$x_n \to x \quad \Longrightarrow \quad Ax_n \to Ax.$$

In applications there are plenty of situations in which a linear operator A may be defined only in a subspace X_0 of a known space, usually a Banach space. In such cases a sequence (x_n) in X_0 may be convergent in X, but the limit need not be in X_0, but the image sequence (Ax_n) can still converge. So, a natural question would be:

> If A is a linear operator defined on a subspace X_0 of a normed linear space X with values in a normed linear space Y, and if (x_n) is a sequence in X_0 such that
>
> $$x_n \to x \text{ in } X \text{ and } Ax_n \to y,$$
>
> then do we have $x \in X_0$ and $y = Ax$?

In view of the question raised above, we have the following definition.

Definition 3.1.1 Let X and Y be normed linear spaces, X_0 be a subspace of X and $A : X_0 \to Y$ be a linear operator. Then A is called a **closed operator** if for every sequence (x_n) is in X_0,

$$x_n \to x \text{ in } X \text{ and } Ax_n \to y \quad \Longrightarrow \quad x \in X_0 \text{ and } y = Ax.$$

♢

We now give a characterization of a closed operator in terms of the closedness of the *graph* of the operator in a *product space*.

Definition 3.1.2 Suppose X and Y are normed linear spaces. Then

$$\|(x, y)\| := \|x\|_X + \|y\|_Y, \quad (x, y) \in X \times Y,$$

defines a norm on the product space $X \times Y$, called the **product norm** on $X \times Y$, and $X \times Y$ with this product norm is called a **product space**. $\Diamond$

Observe:

- For each p with $1 \leq p \leq \infty$,

$$\|(x, y)\| := \begin{cases} (\|x\|_X^p + \|y\|_Y^p)^{1/p}, & 1 \leq p < \infty, \\ \max\{\|x\|_X, \|y\|_Y\}, & p = \infty, \end{cases}$$

Now, the following characterization of a closed operator is immediate.

Proposition 3.1.1 *Let X and Y be normed linear spaces and X_0 be a subspace of X. A linear operator $A : X_0 \to Y$ is a closed operator if and only if its graph,*

$$G(A) := \{(x, Ax) : x \in X_0\},$$

is a closed subset of the product space $X \times Y$.

Example 3.1.1 Let $X = Y = C[a, b]$ with $\|\cdot\|_\infty$ and $X_0 = C^1[a, b]$. Then $A : X_0 \to Y$ defined by

$$Ax = x', \quad x \in X_0$$

is a closed operator: To see this, let (x_n) in X_0 be such that

$$\|x_n - x\|_\infty \to 0 \quad \text{and} \quad \|x_n' - y\|_\infty \to 0$$

for some $x, y \in C[a, b]$, i.e., (x_n) converges to x uniformly and (x_n') converges to y uniformly. Then, by a result in real analysis (see Rudin [9]), we know that x is differentiable and $x' = y$. Thus, $x \in X_0$ and $Ax = y$. $\square$

Example 3.1.2 Let X be an infinite dimensional Hilbert space and $E_0 = \{u_n : n \in \mathbb{N}\}$ be an orthonormal set in X. Let (λ_n) be a sequence of scalars. Let

$$X_0 = \Big\{x \in X : \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, u_n \rangle|^2 < \infty\Big\}.$$

For $x \in X_0$, let

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j.$$

By Riesz-Fischer theorem (Theorem 1.5.4), we see that $Ax \in X$ for every $x \in X_0$, and $A : X_0 \to X$ is a linear operator. We show that

> A is a closed operator, and it is a bounded operator if and only if (λ_n) is a bounded sequence.

Let (x_n) in X_0 be such that

$$x_n \to x \quad \text{and} \quad Ax_n \to y$$

for some $(x, y) \in X \times X$. Then, for each $j \in \mathbb{N}$,

$$\langle x_n, u_j \rangle \to \langle x, u_j \rangle \quad \text{and} \quad \lambda_j \langle x_n, u_j \rangle = \langle Ax_n, u_j \rangle \to \langle y, u_j \rangle.$$

Thus,

$$\lambda_j \langle x, u_j \rangle = \langle y, u_j \rangle \quad \forall\, j \in \mathbb{N}.$$

Hence, $x \in X_0$. Also, if E_0 is an orthonormal basis of X, we obtain $Ax = y$. Suppose E_0 is not an orthonormal basis. Let E be an orthonormal basis which contains E_0. Then for every $u \in E \setminus E_0$, we have $\langle Ax_n, u \rangle = 0$ so that $\langle Ax_n, u \rangle = 0$ for all $n \in \mathbb{N}$ and $\langle y, u \rangle = \lim_{n\to\infty} \langle Ax_n, u \rangle = 0$. Thus,

$$\langle Ax, u \rangle = \langle y, u \rangle \quad \forall u \in E;$$

consequently, $Ax = y$. Thus, we have showed that A is a closed operator.

Clearly, if (λ_n) is bounded sequence with $\beta := \sup_{n\in\mathbb{N}} |\lambda_n|$, then

$$\sum_{j=1}^{\infty} |\lambda_n|^2 |\langle x, u_j \rangle|^2 \le \beta^2 \|x\|^2 \quad \forall\, x \in X,$$

so that $X_0 = X$, $A \in \mathcal{B}(X)$ and $\|A\| \le \beta$. Conversely, if $A \in \mathcal{B}(X)$, then we have

$$|\lambda_n| = \|\lambda_n u_n\| = \|Au_n\| \le \|A\| \quad \forall\, n \in \mathbb{N}$$

so that (λ_n) is bounded. □

The proof of the following theorem is easy and hence it is left as exercise.

Theorem 3.1.2 *Let X and Y be normed linear spaces, X_0 be a subspace of X and $A : X_0 \to Y$ be a closed linear operator. Then the following hold.*

(i) *$N(A)$ is a closed subspace of X.*

(ii) *If A is injective, then $A^{-1} : R(A) \to X$ is a closed operator.*

Is every continuous linear operator a closed operator?

The answer, in general, is not affirmative. Indeed, if X_0 is a non-closed subspace of a normed linear space X then the inclusion operator $I_0 : X_0 \to X$ defined by

$$I_0 x = x \quad \forall\, x \in X_0,$$

is a continuous linear operator, which is not a closed operator. One may also look at the following example which the reader must have seen in real analysis.

Example 3.1.3 Let $X_0 = \mathcal{R}[a,b]$, the space (over $\mathbb{R}$) of all Riemann integrable functions on $[a,b]$, $X = L^1[a,b]$, $Y = \mathbb{R}$, and $A : X_0 \to Y$ be defined by

$$(Ax)(s) = \int_a^b x(t)dt, \quad x \in X_0.$$

Let $\{r_1, r_2, \ldots\}$ be an enumeration of rational numbers in $[a,b]$ and for $n \in \mathbb{N}$, let $x_n : [a,b] \to \mathbb{R}$ be defined by

$$x_n(t) = \begin{cases} 0, & t \in \{r_1, \ldots, r_n\}, \\ 1, & t \notin \{r_1, \ldots, r_n\}. \end{cases}$$

Then, it can be seen easily that $x_n \in \mathcal{R}[a,b]$ and $x_n \to x$ in $L^1[a,b]$, where $x : [a,b] \to \mathbb{R}$ is defined by

$$x(t) = \begin{cases} 0, & t \in \mathbb{Q}, \\ 1, & t \notin \mathbb{Q}. \end{cases}$$

Also, we have

$$\int_a^b x_n(t)dt = b - a \quad \forall\, n \in \mathbb{N}.$$

Thus,

$$x_n \to x \in X \text{ and } Ax_n \to y := b - a;$$

but $x \notin X_0$. □

Theorem 3.1.3 *Let X and Y be normed linear spaces, X_0 be a subspace of X and $A : X_0 \to Y$ be a continuous linear operator.*

(i) *If X_0 is a closed subspace, then A is a closed operator.*

(i) *If Y is a Banach space and A is a closed operator, then X_0 is closed in X.*

Proof. (i) Let (x_n) be a sequence in X_0 such that $x_n \to x$ and $Ax_n \to y$ for some $x \in X$ and $y \in Y$. If X_0 is closed in X, then $x \in X$, so that, by continuity of A, we obtain $Ax_n \to Ax$.

(ii) Suppose Y is a Banach space and A is a closed operator. Let (x_n) be a sequence in X_0 such that $x_n \to x$. By continuity of A, we see that (Ax_n) is a Cauchy sequence in Y. Since Y is complete, there exists $y \in Y$ such that $Ax_n \to y$. Now, by the closedness of A, $x \in X_0$. ∎

The following example illustrates how Theorem 3.1.2(ii) and Theorem 3.1.3 (ii) can also be used to show certain operator is a closed operator.

Example 3.1.4 Let X be an infinite dimensional separable Hilbert space and $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of X. Let (λ_n) be a sequence of nonzero scalars such that

$$d := \inf_{n \in \mathbb{N}} |\lambda_n| > 0.$$

Let A be as in Example 3.1.2. Since $\{u_n : n \in \mathbb{N}\}$ is an orthonormal basis, it follows that, for $x \in X_0$,

$$Ax = 0 \Longrightarrow x = 0$$

so that A is one-one. Further, for every $y \in X$,

$$\sum_{n=1}^{\infty} \frac{|\langle y, u_n \rangle|^2}{|\lambda_n|^2} \le \frac{\|y\|^2}{d^2}$$

so the the series

$$\sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{\lambda_n} u_n$$

converges. Hence, $x := \sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{\lambda_n} u_n$ satisfies the equation $Ax = y$. In other words, A is onto as well. Note that

$$\|A^{-1}y\|^2 = \sum_{n=1}^{\infty} \frac{|\langle y, u_n\rangle|^2}{|\lambda_n|^2} \leq \frac{\|y\|^2}{d^2} \quad \forall\, y \in X.$$

Thus, A^{-1} is continuous with closed domain, the whole of X. By Theorem 3.1.3 (ii), A^{-1} is a closed operator, and hence, by Theorem 3.1.2(ii), A is a closed operator. □

In view of Theorem 3.1.3, a question naturally arises is the following: When is a closed operator continuous? Theorem 2.1.6, together with Theorem 3.1.2(i), shows that every closed linear functional is continuous.

What about if $\dim(Y) > 1$? *Closed graph theorem* gives an answer.

Theorem 3.1.4 (Closed graph theorem) *Let X and Y be Banach spaces and $A : X \to Y$ be a closed operator. Then A is continuous.*

Proof. In order to show that A is continuous, it is enough to show (Why?) that there exists $c > 0$ such that

$$B_0 \subseteq \{x \in X : \|Ax\| \leq c\},$$

where $B_0 = \{x \in X : \|x\| < 1\}$. For $\alpha > 0$, let

$$V_\alpha := \{x \in X : \|Ax\| \leq \alpha\}.$$

Then we have $X = \cup_{j=1}^{\infty} V_j$. Since X is complete, by the Baire category theorem (Theorem 1.3.8), there is some $k \in \mathbb{N}$ such that the interior of $\mathrm{cl}\,(V_k)$ is nonempty. Thus, there is some $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \subseteq \mathrm{cl}\,(V_k)$. Then it can be seen (Verify!) that $B_0 \subseteq \mathrm{cl}\,(V_{2k/r})$. We show that

$$B_0 \subseteq V_{2k/r}. \qquad (*)$$

Let us denote $V_{2k/r}$ by W. Let $x \in B_0$ and $0 < \varepsilon < 1$. Since $B_0 \subseteq \mathrm{cl}\, W$, there exists $x_1 \in W$ such that $\|x - x_1\| < \varepsilon$. Hence, $\varepsilon^{-1}(x - x_1) \in B_0$. By the same argument, there exists $x_2 \in W$ such that $\|\varepsilon^{-1}(x - x_1) - x_2\| < \varepsilon$, i.e.,

$$\|x - (x_1 + \varepsilon x_2)\| < \varepsilon^2\,.$$

Continuing this argument, we obtain a sequence (x_n) in W with

$$\left\|x - (x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots + \varepsilon^{n-1} x_n)\right\| < \varepsilon^n$$

for every positive integer n. In particular, taking $s_n := \sum_{j=1}^n \varepsilon^{j-1} x_j$, $n \in \mathbb{N}$, the sequence (s_n) converges to x. Recall that $x_j \in W$ implies $\|Ax_j\| \le 2k/r$. Hence, for $n > m$, we have

$$\|As_n - As_m\| \le \sum_{j=m+1}^{n} \varepsilon^{j-1} \|Ax_j\| \le \frac{2k}{r} \sum_{j=m+1}^{n} \varepsilon^{j-1}.$$

Thus, (As_n) is a Cauchy sequence in Y. Since Y is also a Banach space, the sequence (As_n) converges to some $y \in Y$. Since A is a closed operator, we have $y = Ax = \lim_{n\to\infty} As_n$. Note that

$$\|As_n\| \le \sum_{j=1}^{n} \varepsilon^{j-1} \|Ax_j\| \le \frac{2k}{r} \sum_{j=1}^{n} \varepsilon^{j-1} \le \frac{2k}{r(1-\varepsilon)}.$$

Hence,

$$\|Ax\| = \lim_{n\to\infty} \|As_n\| \le \frac{2k}{r(1-\varepsilon)}.$$

This is true for all $\varepsilon > 0$. Hence, $\|Ax\| \le 2k/r$. Thus, $(*)$ is proved, which completes the proof. ∎

The following corollary, which is also called *closed graph theorem*, can be deduced from Theorem 3.1.3 and Theorem 3.1.4.

Corollary 3.1.5 (Closed graph theorem) *Let X and Y be Banach spaces and X_0 be a subspace of X. Then a closed operator $A : X_0 \to Y$ is continuous if and only if X_0 is closed in X.*

Here is an application of Theorem 3.1.4.

Theorem 3.1.6 *Let X be a Hilbert space and $A : X \to X$ be a self adjoint operator. Then $A \in \mathcal{B}(X)$.*

Proof. By closed graph theorem, it is enough to prove that A is a closed operator. So, let (x_n) in X be such

$$x_n \to x \quad \text{and} \quad Ax_n \to y$$

for some $x, y \in X$. Using the property of A, we have

$$\langle Ax_n, u\rangle = \langle x_n, Au\rangle \quad \forall\, u \in X.$$

Hence, taking limit as $n \to \infty$, and again using the property of A, we obtain

$$\langle y, u \rangle = \langle x, Au \rangle = \langle Ax, u \rangle \quad \forall\, u \in X.$$

Hence, $y = Ax$. This proves that A is a closed operator. ∎

Now we give examples to show that completeness assumption in Closed Graph Theorem cannot be dropped.

Example 3.1.5 Let $X = C^1[0,1]$ and $Y = C[0,1]$, both with $\|\cdot\|_\infty$ and $X_0 = X$. Let $A : X_0 \to Y$ be defined by

$$Ax = x', \quad x \in X_0.$$

As in Example 3.1.1, we see that A is a closed operator. We have seen in Example 2.1.7 that A is not a continuous operator. Note that X is not a Banach space. □

Example 3.1.6 Let X be an infinite dimensional Banach space and $E := \{u_\lambda : \lambda \in \Lambda\}$ be a basis of X with $\|u_\lambda\| = 1$ for all $\lambda \in \Lambda$. Then E is an uncountable set (Why?). Since E is a basis of X, every $x \in X$ can be written as $x = \sum_{\lambda\in\Lambda} \hat{x}(\lambda) u_\lambda$, where $\hat{x}(\lambda)$ are scalars such that $\hat{x}(\lambda) = 0$ for all but a finite number of λ's. Define

$$\|x\|_* := \sum_{\lambda\in\Lambda} |\hat{x}(\lambda)|, \quad x \in X.$$

Then it can be seen easily that $\|\cdot\|_*$ is also a norm on X and

$$\|x\| \le \|x\|_* \quad \forall\, x \in X.$$

We first show that $\|\cdot\|_*$ is not complete.

Consider a sequence (λ_n) of distinct elements from Λ. For each $n \in \mathbb{N}$, let

$$x_n = \sum_{j=1}^{n} \frac{u_{\lambda_j}}{j^2}.$$

Then, for every $n, m \in \mathbb{N}$ with $n > m$, we have

$$\|x_n - x_m\|_* = \sum_{j=m+1}^{n} \frac{1}{j^2}.$$

Hence, (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_*$. We claim that (x_n) does not converge with respect to $\|\cdot\|_*$. On the contrary, suppose there exists $x \in X$ such that $\|x - x_n\|_* \to 0$ as $n \to \infty$. Then we have

$$\|x - x_n\|_* := \sum_{\lambda \in \Lambda} |\hat{x}(\lambda) - x_n(\lambda)| \geq \left|\hat{x}(\lambda_j) - \frac{1}{j}\right| \quad \forall j \in \mathbb{N}.$$

Since $\|x - x_n\|_* \to 0$, it follows that $\hat{x}(\lambda_j) = 1/j$ for all $j \in \mathbb{N}$, which is not possible. Thus, (x_n) is not convergent with respect to $\|\cdot\|_*$.

Next, let X_* be the linear space X with $\|\cdot\|_*$. Then the identity operator $I : X \to X_*$ is a closed operator. But, it is not continuous, because, if it is is continuous, then there would exist $c > 0$ such that $\|x\|_* \leq c\|x\|$ for all $x \in X$, which would imply that the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent; a contradiction to the fact that $\|\cdot\|$ is complete and $\|\cdot\|_*$ is not complete. □

Now, let us derive some important consequences of closed graph theorem.

3.1.1 Bounded inverse theorem

Theorem 3.1.7 (Bounded inverse theorem) *Suppose X and Y are Banach spaces, X_0 is a subspace of X and $A : X_0 \to Y$ is a closed operator. Suppose A is injective. Then $A^{-1} : R(A) \to X$ is continuous if and only if $R(A)$ is closed.*

Proof. Suppose A is injective. By Theorem 3.1.2, the operator $A^{-1} : R(A) \to X$ is a closed operator. Hence, by Corollary 3.1.5, $A^{-1} : R(A) \to X$ is continuous if and only if $R(A)$ is closed. ∎

The proof of the following corollary, which is also known as *bounded inverse theorem*, is immediate from Theorem 3.1.7.

Corollary 3.1.8 (Bounded inverse theorem) *Suppose X and Y are Banach spaces and $A \in \mathcal{B}(X, Y)$. If A is bijective, then $A^{-1} : Y \to X$ is a bounded operator.*

Here is another consequence of Theorem 3.1.7.

Corollary 3.1.9 *Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are complete norms on a normed linear space X such that one of them is stronger than the other. Then they are equivalent.*

Proof. Suppose $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$, that is, there exists $c > 0$ such that

$$\|x\|_2 \leq c\|x\|_1 \quad \forall\, x \in X.$$

Let X_1 and X_2 be the space X with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. By the above inequality, the identity map from X_1 to X_2 is continuous. Since this map is bijective, and since X_1 and X_2 are Banach spaces, by Corollary 3.1.8, its inverse is also continuous. Hence, there exists $c' > 0$ such that

$$\|x\|_1 \le c'\|x\|_2 \quad \forall\, x \in X.$$

Thus, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ∎

For the next theorem we shall make use of the following lemma.

Lemma 3.1.10 *Let X, Y be normed linear spaces and $A \in \mathcal{B}(X,Y)$. Then $\tilde{A} : X/N(A) \to Y$ defined by*

$$\tilde{A}[x] = Ax, \quad [x] \in X/N(A),$$

is an injective bounded linear operator with $\|\tilde{A}\| = \|A\|$. Further,

$$A \in \mathcal{K}(X,Y) \Longrightarrow \tilde{A} \in \mathcal{K}(X/N(A), Y).$$

Proof. Note that for every $x \in X$,

$$\|\tilde{A}[x]\| = \|A(x-u)\| \le \|A\|\,\|x-u\|, \quad u \in N(A).$$

Hence,

$$\|\tilde{A}[x]\| \le \|A\|\mathrm{dist}(x, N(A)) = \|A\|\,\|[x]\| \quad \forall\, x \in X.$$

so that $\tilde{A}$ is a bounded linear operator and $\|\tilde{A}\| \le \|A\|$. Also, for every $x \in X$,

$$\|Ax\| = \|\tilde{A}[x]\| \le \|\tilde{A}\|\,\|[x]\| \le \|\tilde{A}\|\,\|x\|.$$

Hence, $\|A\| \le \|\tilde{A}\|$. Thus, $\|\tilde{A}\| = \|A\|$.

Next, assume that $A \in \mathcal{K}(X,Y)$. Let (ξ_n) be a bounded sequence in $X/N(A)$, so that there exists $M > 0$ such that $\|\xi_n\| \le M$ for every $n \in \mathbb{N}$. Thus,

$$\mathrm{dist}\,(x_n, N(A)) \le M \quad \forall\, n \in \mathbb{N},$$

where $\xi_n = [x_n]$, $n \in \mathbb{N}$. Hence, there exists a sequence (u_n) in $N(A)$ such that $\|x_n - u_n\| \le 2M$ for all $n \in \mathbb{N}$. In particular, $(x_n - u_n)$ is a bounded sequence in X. Since $A(x_n - u_n) = Ax_n$ for all $n \in \mathbb{N}$ and A is a compact operator, (Ax_n) has a convergent subsequence. But, $\tilde{A}\xi_n = Ax_n$ for all $n \in \mathbb{N}$. Thus, we have proved that $\tilde{A}$ is a compact operator. ∎

Theorem 3.1.11 *Let X and Y be Banach spaces and $A \in \mathcal{K}(X, Y)$. Then $R(A)$ is closed if and only if* $\operatorname{rank}(A) < \infty$.

Proof. Suppose $Y_0 := R(A)$ is closed. Let $\tilde{A} : X/N(A) \to Y_0$ be defined by

$$\tilde{A}[x] = Ax, \quad x \in X.$$

By Lemma 3.1.10, $\tilde{A}$ is a bijective compact operator. Hence, by Theorem 3.1.8, inverse of $\tilde{A}$ is a bounded operator. Therefore, the identity operator on $X/N(A)$ is a compact operator, as it is a composition of a bounded operator with a compact operator. Hence, by Theorem 1.3.4, $X/N(A)$ is finite dimensional; consequently, Y_0 is finite dimensional. ∎

3.1.2 Open mapping theorem

Recall that a function from a metric space to another metric space is said to be an open map if image of every open set is open. In the case of bounded linear operators between Banach spaces we have a nice characterization of open maps. First we prove the following.

Lemma 3.1.12 *Let X be a normed linear space and X_0 be a closed subspace of X. Let $\eta : X \to X/X_0$ be the quotient map, i.e.,*

$$\eta(x) = x + X_0 \quad \forall\, x \in X.$$

Then η is linear, continuous, onto and open.

Proof. Clearly, η is linear and onto . Note that

$$\|\eta(x)\| = dist(x, X_0) \leq \|x\| \quad \forall\, x \in X.$$

Hence, η is continuous. To show that it is open, let G be an open subset of X. We have to show that $\eta(G)$ is open in X/X_0. For this, it is enough to show that for every $x \in G$, there exists $r > 0$ such that

$$y \in X, \quad \|(x + X_0) - (y + X_0)\| < r \Longrightarrow y + X_0 \in \eta(G).$$

So, let $x \in G$. Since G is open, there exists $r > 0$ such that

$$y \in X, \quad \|x - y\| < r \Longrightarrow y \in G.$$

Now, let $y \in X$ be such that $\|(x + X_0) - (y + X_0)\| < r$. Then there exists $u \in X_0$ such that $\|x - y + u\| < r$. Then $y - u \in G$, and hence $y + X_0 = y - u + X_0 \in \eta(G)$. ∎

Theorem 3.1.13 (Open mapping theorem) *Suppose X and Y are Banach spaces and $A : X \to Y$ is a bounded linear operator. Then A is an open map if and only if it is onto.*

Proof. Suppose A is an open map, i.e., A maps every open subset of X onto an open subset of Y. In particular, $R(A)$, the image of the open set X is a nonempty open subset of Y. By Theorem 1.3.9, this is possible only if $R(A) = Y$, i.e., A is onto.

Conversely, suppose that A is onto. Then the linear operator $\widetilde{A} : X/N(A) \to Y$ defined by

$$\widetilde{A}[x] = Ax, \quad [x] \in X/N(A),$$

is a bijective bounded linear operator between Banach spaces $X/N(A)$ and Y (cf. Lemma 3.1.10). Hence, by Bounded Inverse Theorem, inverse of $\widetilde{A}$ is also continuous. In particular, $\widetilde{A}$ is an open map. Since

$$A = \eta \circ \widetilde{A},$$

where $\eta : X \to X/N(A)$ is the quotient map as in Lemma 3.1.12, we obtain that A is also an open map. ∎

3.1.3 Uniform boundedness principle

From analysis we know that if a sequence (f_n) of real valued continuous functions defined on a metric space Ω converges uniformly to a function $f : \Omega \to \mathbb{R}$, then f is also continuous. However, if the uniform convergence is replaced by pointwise convergence, i.e.,

$$f_n(x) \to f(x) \quad \text{as} \quad n \to \infty \quad \text{for each} \quad x \in \Omega,$$

then the function f need not be continuous. This is the case even for continuous linear functionals on a normed linear space. However, if the domain space is a Banach space, the the limiting functional is continuous. We shall derive this fact as a consequence of the following general result.

Theorem 3.1.14 (Uniform boundedness principle) *Let X be a Banach space, Y be a normed linear space and $\mathcal{A}$ be subset of $\mathcal{B}(X, Y)$ such that $\{Ax : A \in \mathcal{A}\}$ is bounded for each $x \in X$. Then $\mathcal{A}$ is a bounded subset of $\mathcal{B}(X, Y)$.*

Proof. Let us denote the norms on X and Y by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Since $\{Ax : A \in \mathcal{A}\}$ is bounded in X, for each $x \in X$, $\sup_{A\in\mathcal{A}} \|Ax\|_Y$ is a well defined non-negative real number. Define

$$\|x\|_* := \|x\|_X + \sup_{A\in\mathcal{A}} \|Ax\|_Y, \quad x \in X.$$

It is easily seen that $\|\cdot\|_*$ is also a norm on X. Further, $\|\cdot\|_*$ is stronger than $\|\cdot\|_X$. Now, we show that $\|\cdot\|_*$ is complete. For this, let (x_n) be a Cauchy sequence in X with respect to $\|\cdot\|_*$. Since $\|\cdot\|_*$ is stronger than $\|\cdot\|_X$, (x_n) is a Cauchy sequence in X with respect to $\|\cdot\|_X$ as well. Using the completeness of $\|\cdot\|_X$, there exists $x \in X$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$. Hence, for every $A \in \mathcal{A}$, by its continuity, $\|Ax_n - Ax\| \to 0$ as $n \to \infty$. Now, let $\varepsilon > 0$ be given. Since (x_n) is a Cauchy sequence in X with respect to $\|\cdot\|_*$, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_X + \sup_{A\in\mathcal{A}} \|Ax_n - Ax_m\|_Y < \varepsilon \quad \forall\, n \geq N.$$

Let $A \in \mathcal{A}$. From the above inequality, we have

$$\|x_n - x_m\|_X + \|Ax_n - Ax_m\|_Y < \varepsilon \quad \forall\, n \geq N, \tag{i}$$

and since, $\|Ax_n - Ax\|_X \to 0$ as $n \to \infty$, letting $m \to \infty$ in (i), we have

$$\|x_n - x\|_X + \|Ax_n - Ax\|_Y \leq \varepsilon \quad \forall\, n \geq N.$$

This is true for every $A \in \mathcal{A}$. Therefore,

$$\|x_n - x\|_* = \|x_n - x\|_X + \sup_{A\in\mathcal{A}} \|Ax_n - Ax\|_Y \leq \varepsilon \quad \forall\, n \geq N.$$

Thus, we have shown that (x_n) converges with respect to $\|\cdot\|_*$, and consequently, $\|\cdot\|_*$ is complete. Hence, by Corollary 3.1.9, $\|\cdot\|_*$ and $\|\cdot\|_X$ are equivalent, so that there exists $c > 0$ such that

$$\|x\|_* \leq c\|x\|_X \quad \forall\, x \in X.$$

In particular,

$$\sup_{A\in\mathcal{A}} \|Ax\|_Y \leq c\|x\|_X \quad \forall\, x \in X.$$

Hence, $\|A\| \leq c$ for all $A \in \mathcal{A}$, and the proof is complete. ∎

Remark 3.1.1 The proof of Uniform Boundedness Principle given above is different from that usually appear in standard text books on Functional Analysis. This proof was conveyed to the author by Professor S. Ramaswamy [7]. ◊

Corollary 3.1.15 (Banach–Steinhaus theorem) *Let X be a Banach space, Y be a normed linear space and (A_n) be a sequence of operators in $\mathcal{B}(X,Y)$ which converges pointwise on X. Then $(\|A_n\|)$ is bounded and the operator $A : X \to Y$ defined by*

$$Ax := \lim_{n\to\infty} A_n x, \quad x \in X,$$

belongs to $\mathcal{B}(X,Y)$.

Proof. It can seen easily that A is a linear operator. Now, since (A_n) converges pointwise on X, by Theorem 3.1.14, $(\|A_n\|)$ is bounded, say $\|A_n\| \le M$ for all $n \in \mathbb{N}$ for some $M > 0$.

Now, let $x \in S$ be such that $\|x\| \le 1$. Let $N \in \mathbb{N}$ be such that $\|A_n x - Ax\| < 1$ for all $n \ge N$. Then we have

$$\|Ax\| \le \|Ax - A_N x\| + \|A_N x\| \le 1 + M.$$

This is true for all $x \in S$. Hence, $A \in \mathcal{B}(X,Y)$. ∎

3.2 Hahn-Banach Extension Theorem

We know that if X is a Hilbert space, then its dual can be identified with X by a conjugate linear isometry.

What can we say about the dual of a general normed linear space?

Of course, if X is finite dimensional, then we know that X' is of the same dimension as that of X. Also, in certain specific cases, we can identify the dual space. In this context, we may recall from Section 2.3 the following:

- For $1 \le p < \infty$, the dual of ℓ^p is linearly isometric with ℓ^q.
- For $1 \le p < \infty$, the dual of $L^p[a,b]$ is linearly isometric with $L^q[a,b]$.
- The dual of $C[a,b]$ with $\|\cdot\|_\infty$ is linearly isometric with $NBV[a,b]$.

Here q is the conjugate exponent of p, i.e., $1/p + 1/q = 1$. However, using the theory discussed so far, we are not in a position to say even that X' is nonzero whenever X is nonzero! Our attempt is to prove a general theorem, called *Hahn-Banach extension theorem*, using which we shall, in fact, show that

$$\dim(X') \geq \dim(X).$$

3.2.1 The theorem and its consequences

Theorem 3.2.1 (Hahn-Banach extension theorem) (HBET) *Let X_0 be a subspace of a normed linear space X. If $f_0 \in X_0'$, then there exists $f \in X'$ such that*

$$f_{|X_0} = f_0 \quad \textit{and} \quad \|f\| = \|f_0\|.$$

Before proving HBET, let us deduce some of its consequences.

Corollary 3.2.2 *Let X be a nonzero normed linear space and x_0 be a nonzero element in X. Then there exists $f \in X'$ such that*

$$f(x_0) = \|x_0\| \quad \textit{and} \quad \|f\| = 1.$$

Proof. Let $X_0 = \operatorname{span}\{x_0\}$, and define $f_0 : X_0 \to \mathbb{K}$ by

$$f_0(\alpha x_0) = \alpha\|x_0\|, \quad \alpha \in \mathbb{K}.$$

Clearly, f_0 is a linear functional on X_0. Further, $f_0 \in X_0'$ and $\|f_0\| = 1$ (Exercise). Hence, by HBET, there exists $f \in X'$ such that $f(x_0) = f_0(x_0) = \|x_0\|$ and $\|f\| = \|f_0\| = 1$. ∎

More generally we have the following.

Corollary 3.2.3 *Let X_0 be a closed proper subspace of a normed linear space X and $x_0 \in X \setminus X_0$. Then there exists $f \in X'$ such that*

$$f(x_0) = \operatorname{dist}(x_0, X_0) \quad \|f\| = 1 \quad \textit{and} \quad f_{|X_0} = 0.$$

Proof. Let $X_1 = \operatorname{span}\{x_0, X_0\}$, and define $f_0 : X_1 \to \mathbb{K}$ by

$$f_0(\alpha x_0 + u) = \alpha \operatorname{dist}(x_0, X_0), \quad \alpha \in \mathbb{K}, \quad u \in X_0.$$

Clearly, f_0 is a linear functional on X_1. Further,

$$|f_0(\alpha x_0 + u)| = \operatorname{dist}(\alpha x_0, X_0) = \operatorname{dist}(\alpha x_0 + u, X_0) \leq \|\alpha x_0 + u\|$$

for all $\alpha \in \mathbb{K}$, $u \in X_0$. Thus, $f_0 \in X_1'$ and $\|f_0\| \leq 1$. Also, we have

$$\text{dist}\,(x_0, X_0) = |f_0(x_0)| = |f_0(x_0 - u)| \leq \|f_0\|\|x_0 - u\| \quad \forall\, u \in X_0.$$

Hence,

$$\text{dist}\,(x_0, X_0) \leq \|f_0\|\text{dist}\,(x_0, X_0).$$

Since $\text{dist}\,(x_0, X_0) > 0$, we have $\|f_0\| \geq 1$. Thus, $\|f_0\| = 1$. Hence, by HBET, there exists $f \in X'$ satisfying

$$f_{|X_1} = f_0 \quad \text{and} \quad \|f\| = \|f_0\|.$$

In particular,

$$f_{|X_0} = f_{0|X_0} = 0, \quad \|f\| = \|f_0\| = 1$$

and

$$f(x_0) = f_0(x_0) = \text{dist}\,(x_0, X_0).$$

This completes the proof. ∎

An immediate consequence of the above corollary, which is often used in applications, is the following.

Corollary 3.2.4 *Let X_0 be a subspace of a normed linear space X. If there exists a nonzero $f \in X'$ such that $f(x) = 0$ for every $x \in X_0$, then X_0 is not dense in X.*

Corollary 3.2.5 *Let X be a normed linear space and $\{u_1, \ldots, u_k\}$ be a linearly independent subset of X. Then there exists a linearly independent set $\{f_1, \ldots, f_k\} \subset X'$ such that*

$$f_i(u_j) = \delta_{ij}, \quad i, j = 1, \ldots, k.$$

Proof. Let $X_0 = \text{span}\,\{u_1, \ldots, u_k\}$. Then, from linear algebra, we know that there exist linear functionals g_i on X_0 such that

$$g_i(u_j) = \delta_{ij} \quad \forall\, i, j = 1, \ldots, k.$$

In fact, g_i is defined by

$$g_i\Big(\sum_{j=1}^{k} \alpha_j u_j\Big) = \alpha_i, \quad \alpha_i \in \mathbb{K}.$$

Since X_0 is finite dimensional, by Theorem 2.1.4, $g_i \in X_0'$. Hence, by Theorem 3.2.1, each g_i has a norm preserving extension f_i to all of X, so that $f_i \in X'$ and

$$f_i(u_j) = g_i(u_j) = \delta_{ij}, \quad i, j = 1, \ldots, k.$$

This completes the proof. ∎

Corollary 3.2.6 *Let X be a normed linear space and X_0 be a finite dimensional subspace of X. Then there exists a closed subspace Z of X such that $X = X_0 + Z$ and $X_0 \cap Z = \{0\}$.*

Proof. Let $\dim(X_0) = n$ and let $\{u_1, \ldots, u_n\}$ be a basis of X_0. Let $f_1, \ldots, f_k$ be as in Corollary 3.2.5. Then every $x \in X$ can be expressed as $x = y + z$ where

$$y = \sum_{j=1}^{n} f_j(x)u_j \in X_0 \quad \text{and} \quad z = x - y \in Z := \bigcap_{j=1}^{n} N(f_j).$$

Note that Z is a closed subspace of X and $X_0 \cap Z = \{0\}$. ∎

Corollary 3.2.7 *Let $A : X \to Y$ be a finite rank linear operator between normed linear spaces X and Y. Then $A \in \mathcal{B}(X, Y)$ if and only if there exist $y_1, \ldots, y_n$ in Y and continuous linear functionals $f_1, \ldots, f_n$ on X such that*

$$Ax = \sum_{i=1}^{n} f_i(x)y_i, \quad x \in X.$$

Proof. Let $A \in \mathcal{B}(X, Y)$ be of finite rank, say $\text{rank}(A) = k$, and let $\{y_1, \ldots, y_k\}$ be a basis of $R(A)$. Then, for every $x \in X$, there exist scalars $\alpha_1(x), \ldots, \alpha_k(x)$ such that

$$Ax = \sum_{j=1}^{k} \alpha_j(x)y_j. \tag{$*$}$$

Now, by Corollary 3.2.5, there exist continuous linear functionals $g_1, \ldots, g_k$ on $R(A)$ such that $g_i(y_j) = \delta_{ij}$. Thus, from $(*)$, we have

$$g_i(Ax) = \sum_{j=1}^{k} \alpha_j(x)g_i(y_j) = \alpha_i(x) \quad \forall\, x \in X.$$

Thus, taking $f_i = g_i \circ A$ we see that $f_i \in X'$ and

$$Ax = \sum_{j=1}^{k} f_j(x)(x)y_j. \quad \forall\, x \in X.$$

The converse part is obvious. ∎

3.2.2 Proof of the theorem

We shall prove a theorem in a slightly general context and derive Theorem 3.2.1 as a corollary to that. First a definition.

Definition 3.2.1 Let X be a linear space over $\mathbb{C}$.

(i) A linear functional $f : X \to \mathbb{C}$ is called a **complex-linear functional**.

(ii) A function $f : X \to \mathbb{R}$ is called a **real-linear functional** if f is a linear functional considering X as a linear space over $\mathbb{R}$, i.e., if

$$f(x + y) = f(x) + f(y), \quad f(\alpha x) = \alpha f(x)$$

for all x, y in X and $\alpha \in \mathbb{R}$. $\Diamond$

We shall also make use of the following two lemmas.

Lemma 3.2.8 *Let X be a linear space over $\mathbb{C}$.*

(i) Let $f : X \to \mathbb{C}$ be a complex-linear functional. Then the function $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(x) = \mathrm{Re} f(x), \quad x \in X,$$

is a real-linear functional and $f(x) = \varphi(x) - i\varphi(ix)$ for all $x \in X$.

(ii) Let $\varphi : X \to \mathbb{R}$ be a real-linear functional. Then the function $f : X \to \mathbb{C}$ defined by

$$f(x) = \varphi(x) - i\varphi(ix), \quad x \in X,$$

is a complex-linear functional.

Proof. (i) It can be easily seen that

$$\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(\alpha x) = \alpha\varphi(x)$$

for all x, y in X and $\alpha \in \mathbb{R}$. Thus, φ is a real-linear functional. Next, let

$$\psi(x) = \mathrm{Im}\, f(x), \quad x \in X.$$

Then we have $f(x) = \varphi(x) + i\psi(x)$ for all $x \in X$. Also, for all $x \in X$, since $f(ix) = if(x)$, we have

$$\varphi(ix) + i\psi(ix) = -\psi(x) + i\varphi(x).$$

Therefore, $\psi(x) = -\varphi(ix)$ so that

$$f(x) = \varphi(x) + i\psi(x) = \varphi(x) - i\varphi(ix) \quad \forall\, x \in X.$$

(ii) It can be easily seen that

$$f(x+y) = f(x) + f(y), \quad f(\alpha x) = \alpha f(x)$$

for all x, y in X and $\alpha \in \mathbb{R}$. Also, for $x \in X$, we have

$$\begin{aligned} f(ix) &= \varphi(ix) - i\varphi(-x) \\ &= \varphi(ix) + i\varphi(x) \\ &= i[\varphi(x) - i\varphi(ix)] \\ &= if(x). \end{aligned}$$

Hence, for $x \in X$ and α, β in $\mathbb{R}$,

$$f(\alpha x + i\beta x) = f(\alpha x) + f(i\beta x) = \alpha f(x) + \beta f(ix) = \alpha f(x) + i\beta f(x).$$

Thus, for $x \in X$ and $\lambda \in \mathbb{C}$, we have

$$f(\lambda x) = \lambda f(x).$$

This completes the proof. ∎

Lemma 3.2.9 *Let X be a linear space over $\mathbb{C}$, $p : X \to \mathbb{R}$ be a seminorm and $f : X \to \mathbb{C}$ be a linear functional. Then*

$$|f(x)| \le p(x) \quad \forall\, x \in X \iff |\mathrm{Re} f(x)| \le p(x) \quad \forall\, x \in X.$$

Proof. Clearly,

$$|f(x)| \le p(x) \quad \forall\, x \in X \Longrightarrow |\mathrm{Re} f(x)| \le p(x) \quad \forall\, x \in X.$$

Conversely, suppose $|\mathrm{Re} f(x)| \le p(x)$ for all $x \in X$. Now, if $x \in X$, then $|f(x)| = \lambda f(x)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and . Thus,

$$|f(x)| = \lambda f(x) = f(\lambda x)$$

so that $|f(x)| = \mathrm{Re} f(\lambda x)$ and hence

$$|f(x)| = |\mathrm{Re} f(\lambda x)| \le p(\lambda x) = |\lambda| p(x) = p(x).$$

This completes the proof. ∎

We shall derive Theorem 3.2.1 from the following general version.

Theorem 3.2.10 (Hahn-Banach extension theorem) (HBET) *Let X_0 be a subspace of a linear space X and $p : X \to \mathbb{R}$ be a seminorm. If $g : X_0 \to \mathbb{K}$ is a linear functional on X_0 such that*

$$|g(x)| \le p(x) \quad \forall\, x \in X_0$$

then there exists a linear functional $f : X \to \mathbb{K}$ on X such that

$$|f(x)| \le p(x) \quad \forall\, x \in X.$$

Proof. If $X_0 = X$ or $g = 0$, then we can take $f = g$. So assume that $X_0 \ne X$ and $g \ne 0$.

First we consider the case of $\mathbb{K} = \mathbb{R}$.

Let $x_0 \in X \setminus X_0$. The idea of the proof is that first we extend g to a linear functional on span $\{X_0, x_0\}$ satisfying the requirements and then use that result to extend to all of X. So, let

$$\widetilde{X}_0 := \text{span}\,\{x_0; X_0\} = \{u + \alpha x_0 : u \in X_0,\ \alpha \in \mathbb{R}\}.$$

Note that for every $u, v \in X_0$,

$$g(u) - g(v) = g(u - v) \le p(u - v) \le p(u - x_0) + p(v - x_0)$$

so that

$$g(u) - p(u - x_0) \le g(v) + p(v - x_0) \qquad \forall\, u, v \in X_0.$$

Hence,

$$\sup\,\{g(u) - p(u - x_0) : u \in X_0\} \le \inf\,\{g(u) + p(u - x_0) : u \in X_0\}.$$

Now, let $r \in \mathbb{R}$ be such that

$$\sup\,\{g(u) - p(u - x_0) : u \in X_0\} \le r \le \inf\,\{g(u) + p(u - x_0) : u \in X_0\}.$$

Then we have

$$g(u) - p(u - x_0) \le r \quad \forall\, u \in X_0, \quad r \le g(u) + p(u - x_0) \quad \forall\, u \in X_0,$$

so that

$$|g(u) - r| \le p(u - x_0) \qquad \forall\, u \in X_0.$$

Let $\tilde{g} : \widetilde{X}_0 \to \mathbb{R}$ be defined by

$$\tilde{g}(u + \alpha x_0) = g(u) + \alpha r, \qquad u \in X_0,\ \alpha \in \mathbb{R}.$$

Then, it can be easily verified that $\tilde{g}$ is a linear functional on $\widetilde{X}_0$. Further, for $u \in X_0$ and $\alpha \neq 0$, we obtain

$$\begin{aligned} |g(u) + \alpha r| &= |(-\alpha)[g(-u/\alpha) - r]| \\ &\leq |\alpha| p(-u/\alpha - x_0) \\ &= p(u + \alpha x_0). \end{aligned}$$

Thus,

$$|\tilde{g}(u + \alpha x_0)| \leq p(u + \alpha x_0) \qquad \forall\, u \in X_0,\ \alpha \in \mathbb{R}.$$

Thus, we have proved that $\tilde{g} : \widetilde{X}_0 \to \mathbb{R}$ is a linear functional satisfying

$$|\tilde{g}(x)| \leq p(x) \qquad \forall\, x \in \widetilde{X}_0.$$

We shall use the above result, along with Zorn's lemma, to obtain a linear extension $f : X \to \mathbb{R}$ of g such that $|f(x)| \leq p(x)$ for every $x \in X$. For this purpose, consider the family $\mathcal{S}$ of all pairs (Y, h), where Y is a subspace of X such that $X_0 \subseteq Y$ and $h : Y \to \mathbb{R}$ is a linear extension of g such that $|h(x)| \leq p(x)$ for all $x \in Y$. This family $\mathcal{S}$ is non-empty, since $(\widetilde{X}_0, \tilde{g})$ obtained in the last paragraph belongs to $\mathcal{S}$. For (Y_1, h_1), (Y_2, h_2) in $\mathcal{S}$, define $(Y_1, h_1) \preccurlyeq (Y_2, h_2)$ whenever $Y_1 \subseteq Y_2$ and h_2 is an extension of h_1. It can be seen that $\preccurlyeq$ is a partial order on $\mathcal{S}$. Suppose $\mathcal{T}$ is a totally ordered subset of $\mathcal{S}$. Then consider

$$Z = \cup\{Y : (Y, h) \in \mathcal{T}\},$$

and define $\phi : Z \to \mathbb{R}$ such that $\phi(x) = h(x)$ whenever $x \in Y$, $(Y, h) \in \mathcal{T}$. Then, we see that $(Z, \phi) \in \mathcal{S}$, and (Z, ϕ) is an upper bound of $\mathcal{T}$. Therefore, by Zorn's lemma, $\mathcal{S}$ has a maximal element, say (Y_0, f). Now, we show that $Y_0 = X$.

Suppose $Y_0 \neq X$, and let $y_0 \in X \setminus Y_0$. Then, by the first part of the proof, f has a linear extension, say $\tilde{f}$ to $\widetilde{Y}_0 := \operatorname{span}\{y_0; Y_0\}$ satisfying $|\tilde{f}(x)| \leq p(x)$ for all $x \in \widetilde{Y}_0$. Thus, we have

$$(Y_0, f) \preccurlyeq (\widetilde{Y}_0, \tilde{f}) \in \mathcal{S}, \qquad (Y_0, f) \neq (\widetilde{Y}_0, \tilde{f})$$

contradicting the maximality of (Y_0, f). Therefore, $Y_0 = X$, and f is a linear extension of g satisfying $|f(x)| \leq p(x)$ for all $x \in X$.

Now we take up the case of $\mathbb{K} = \mathbb{C}$. Let $g_0 : X_0 \to \mathbb{R}$ be defined by

$$g_0(x) = \operatorname{Re} g(x), \quad x \in X_0.$$

Then by Lemma 3.2.8 and Lemma 3.2.9, g_0 is a real-linear functional on X_0 satisfying $|g_0(x)| \le \nu(x)$ for all $x \in X_0$. Therefore, by the earlier part, g_0 has a real-linear extension $f_0 : X \to \mathbb{R}$ satisfying $|f_0(x)| \le \nu(x)$ for all $x \in X$. Again, by Lemma 3.2.8 and Lemma 3.2.9, the function $f : X \to \mathbb{C}$ defined by

$$f(x) = f_0(x) - i f_0(ix), \qquad x \in X,$$

is a complex-linear functional satisfying $|f(x)| \le \nu(x)$ for all $x \in X$. This f is an extension of g since, for every $x \in X_0$,

$$f(x) = f_0(x) - i f_0(ix) = g_0(x) - i g_0(ix) = g(x).$$

This completes the proof of the theorem. ∎

Remark 3.2.1 A natural question that may come into mind is whether there is an analogue for Hahn-Banach extension theorem for general operators. In general, the answer is not in affirmative. To see this suppose X is an infinite dimensional normed linear space and X_0 is a non-closed subspace of X. Let $A_0 : X_0 \to X_0$ be the identity operator on X_0, i.e.,

$$A_0 x = x \quad \forall\, x \in X_0.$$

Clearly, $A_0 \in \mathcal{B}(X_0)$ with $\|A_0\| = 1$. However, there is no norm preserving extension for A_0 to all of X. To see this, suppose there is a norm preserving extension $A : X \to X_0$. Then taking $I_0 : X_0 \to X$ as the inclusion operator, we see that

$$P := I_0 A$$

is a projection operator in $\mathcal{B}(X)$ with $\|P\| = 1$ and $R(P) = X_0$. This forces X_0 to be a closed subspace, which is a contradiction to the assumption on X_0. ♢

3.2.3 Further consequences

We have seen that every normed linear space is linearly isometric with a dense subspace of a Banach space (cf. Theorem 1.3.16). This fact is also a consequence of Hahn-Banach extension theorem, as the following theorem shows.

Theorem 3.2.11 *Let X be a normed linear space. For each $x \in X$, let $\hat{x} : X' \to \mathbb{K}$ be defined by*

$$\hat{x}(f) = f(x), \quad f \in X'.$$

Then $\hat{x} \in X''$ for every $x \in X$, and the function $J : X \to X''$ defined by

$$J(x) = \hat{x}, \quad x \in X,$$

is a linear isometry. In particular, closure of $\{\hat{x} : x \in X\}$ is a completion of X.

Proof. Let $x \in X$. Then we observe that

$$\hat{x}(f + \alpha g) = (f + \alpha g)(x) = f(x) + \alpha g(x) = \hat{x}(f) + \alpha \hat{x}(g)$$

for all $f, g \in X$ and for all $\alpha \in \mathbb{K}$. Further,

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \, \|x\| \quad \forall \, f \in X'.$$

Hence, $\hat{x} \in X''$ for each $x \in X$ and $\|\hat{x}\| \leq \|x\|$. Clearly, if $x = 0$, then $\hat{x} = 0$. If $x \neq 0$, then by Corollary 3.2.2, there exists $f_x \in X'$ such that

$$f_x(x) = \|x\| \quad \text{and} \quad \|f_x\| = 1.$$

Thus,

$$\|x\| = |f_x(x)| = |\hat{x}(f_x)| \leq \|\hat{x}\| \, \|f_x\| = \|\hat{x}\|.$$

Thus, we have proved that

$$\|\hat{x}\| = \|x\| \quad \forall \, x \in X.$$

It remains to show that $J : x \mapsto \hat{x}$ is a linear operator. For this, let $x, y \in X$ and $\alpha \in \mathbb{K}$. Then, for every $f \in X'$, we have

$$\begin{aligned} \varphi_{x+\alpha t}(f) &= f(x + \alpha y) = f(x) + \alpha f(y) \\ &= \hat{x}(f) + \alpha \varphi_y(f) = (\hat{x} + \alpha \varphi_y)(f). \end{aligned}$$

The last part follows, because, X'' is a Banach space. This competes the proof. ∎

Definition 3.2.2 Let X be a normed linear space.

1. The linear isometry $J : X \to X''$ obtained in Theorem 3.2.11 is called the **canonical isometry** from X to X''.

2. The space X is said to be a **reflexive space** if the canonical isometry $J : X \to X''$ is surjective.

$\Diamond$

Clearly, a reflexive space has to be a Banach space. It is known (cf. Nair [5]) that the spaces

- ℓ^p and $L^p[a,b]$ for $1 < p < \infty$, and Hilbert spaces are reflexive spaces,

whereas the spaces

- ℓ^1, ℓ^∞, $L^1[a,b]$, $L^\infty[a,b]$ and $C[a,b]$ (with $\|\cdot\|_\infty$) are not reflexive spaces.

Theorem 3.2.12 *Let X be a non-zero normed linear space and let*

$$\Omega := \{f \in X' : \|f\| = 1\}.$$

For each $x \in X$, let $\varphi_x : \Omega \to \mathbb{K}$ be defined by

$$\varphi_x(f) = f(x), \quad f \in \Omega.$$

Then $\varphi_x \in C_b(\Omega)$ for every $x \in X$, and the function $T : X \to C_b(\Omega)$ defined by

$$T(x) = \varphi_x, \quad x \in X,$$

is a linear isometry. In particular,

$$\widehat{X} := \mathrm{cl}\,\{T(x) : x \in X\}$$

is a closed subspace of $C_b(\Omega)$ and it is a completion of X.

Proof. Use the arguments as in Theorem 3.2.11. ∎

Note that, for proving the last part of the Theorem 3.2.11, we used the fact that dual of a normed linear space is a Banach space (cf. Theorem 2.1.3). Now, we prove the converse of this statement.

Theorem 3.2.13 *Let X and Y be normed linear spaces with $X \neq \{0\}$ and let $\mathcal{B}(X,Y)$ be a Banach space. Then Y is a Banach space.*

Proof. Let (y_n) be a Cauchy sequence in Y. We have to show that (y_n) converges to some element in Y. Since $X \neq \{0\}$, there exists $x_0 \in X$ such that $\|x_0\| = 1$. By Corollary 3.2.2, there exists $f_0 \in X'$ be such that

$$f_0(x_0) = \|x_0\| = 1 \quad \text{and} \quad \|f_0\| = 1.$$

For each $n \in \mathbb{N}$, let $A_n : X \to Y$ be defined by

$$A_n x = f_0(x) y_n, \quad x \in X.$$

Clearly, A_n is a linear operator for every $n \in \mathbb{N}$. Further, for every $x \in X$ and $n \in \mathbb{N}$, we have

$$\|A_n x\| = \|f_0(x) y_n\| = |f_0(x)|\, \|y_n\| \leq \|y_n\| \|x\|$$

so that $A_n \in \mathcal{B}(X, Y)$ and $\|A_n\| \leq \|y_n\|$ for all $n \in \mathbb{N}$. Also, we have, for every $x \in X$ and $n \in \mathbb{N}$,

$$\|(A_n - A_m)x\| = \|A_n x - A_m x\| = \|f_0(x)(y_n - y_m)\| \leq \|y_n - y_m\| \|x\|$$

so that $\|A_n - A_m\| \leq \|y_n - y_m\|$ for all $n \in \mathbb{N}$. Consequently, (A_n) is a Cauchy sequence in $\mathcal{B}(X, Y)$. Since $\mathcal{B}(X, Y)$ is Banach space, there exists $A \in \mathcal{B}(X, Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$. Thus, in particular, taking $y_0 := A x_0$, we have

$$\|y_n - y_0\| = \|A_n x_0 - A x_0\| \to 0 \quad \text{as} \quad n \to \infty.$$

This competes the proof. ∎

3.2.4 Problems

1. Let X and Y be inner product spaces and $A : X \to Y$ be a linear operator. Prove that, if A^* exists then A^* is a closed operator.

2. Every self adjoint operator on an inner product space is a closed operator. Why?

3. Let X be an inner product space, Y be Hilbert space and X_0 be a dense subspace of X. Let $A : X_0 \to Y$ be a linear operator. Prove that there exists a subspace Y_0 of Y and a closed operator $B : Y_0 \to X$ such that

$$\langle Ax, y \rangle = \langle x, By \rangle \quad \forall\, x \in X_0,\, y \in Y_0.$$

4. Every linear functional on a normed linear space which is a closed operator is continuous. Why?

5. Is every continuous linear functional on a normed linear space a closed operator? Why?

6. Let X be a Banach space, Y be a normed linear space, X_0 be a subspace of X and $A : X_0 \to Y$ be a closed operator. Prove that if A is bounded below, then $R(A)$ is a closed subspace.

7. Let X be a Banach space, X_0 be a subspace of X and $A : X_0 \to Y$ be a linear operator which is bounded below. Prove that A is a closed operator if and only if $R(A)$ is closed.

8. Let $X = L^2[0,1] = Y$ and $Ax = x'$, $x \in X_0$, where X_0 is linear space of all $x \in L^2[0,1]$ such that x is absolutely continuous with $x(0) = 0$ and $x' \in L^2[0,1]\}$. Prove that $A : X_0 \to L^2[a,b]$ is a bijective, closed operator.

9. Let X be a Banach space, Y be a normed linear space, X_0 be a subspace of X and $A : X_0 \to Y$ be a closed operator. Prove that the following are equivalent:

   (a) A is bijective and A^{-1} is continuous,

   (b) A is bounded below and $R(A)$ is dense.

10. Let X be a Banach space and $P \in \mathcal{B}(X)$ be a projection operator. Prove that $P \in \mathcal{K}(X)$ if and only if $\operatorname{rank}(P) < \infty$.

11. Let X be a Banach space and $P : X \to X$ be a projection operator. Prove that $P \in \mathcal{B}(X)$ if and only if $R(P)$ and $N(P)$ are closed subspaces of X.

12. Let X and Y be a Banach spaces, and (A_n) be a sequence of operators in $\mathcal{B}(X,Y)$ which converges pointwise on a dense subset of $\{x \in X : \|x\| \le 1\}$. Prove that, if S is a relatively bounded subset of X, then

$$\sup\{\|A_n x - Ax\| : x \in S\} \to 0 \quad \text{as} \quad n \to \infty,$$

that is, (A_n) converges to A uniformly on S.

13. Let X be a Banach space and (P_n) in $\mathcal{B}(X)$ be a sequence of finite rank projection operators which converges pointwise. Prove that $P : X \to X$ defined by $Px = \lim_{n\to\infty} P_n x$ is a projection operator and $P \in \mathcal{B}(X)$.

14. Let X be a normed linear space and (x_n) be a sequence in X such that, there is an $x \in X$ satisfying $f(x_n) \to f(x)$ for every $f \in X$. Prove that (x_n) is a bounded sequence. Is it necessary that (x_n) converges to x? Why?

15. Let X be a normed linear space. Prove that if $x, y \in X$ with $x \neq y$, then there exists $f \in X'$ such that $f(x) \neq f(y)$.

16. Prove Corollary 3.2.4.

17. Let X be a normed linear space, and for a subset S of X, let

$$S^a := \{f \in X' : f(x) = 0 \quad \forall\, x \in S\}.$$

Prove the following:

(a) S^a is a closed subspace of X.

(b) If S is a subspace which is not dense in X, then $S^a \neq \{0\}$.

18. Let X be a normed linear space and X_0 be a finite dimensional subspace of X. Prove that there exists a projection operator $P \in \mathcal{B}(X)$ such that $R(P) = X_0$.

19. Give details of the proof of Theorem 3.2.12.