4

Spectral Results

4.1 Eigen Spectrum and Approximate Eigen Spectrum

Let X be a linear space and $A : X \to X$ be a linear operator. We recall the following definition from linear algebra.

Definition 4.1.1 A scalar λ is an **eigenvalue** of A if there exists a nonzero $x \in X$ such that

$$Ax = \lambda x,$$

and in that case x is called an **eigenvector** of A corresponding to the eigenvalue λ. The set of all eigenvalues of A is called the **eigen spectrum** of A and it is denoted by $\sigma_{\mathrm{eig}}(A)$. $\Diamond$

Thus, λ is an eigenvalue of A if and only if $A - \lambda I$ is not one–one.

- λ is an eigenvalue of A if and only if $N(A - \lambda I)$ is non-trivial, and in that case every nonzero vector in $N(A-\lambda I)$ is an eigenvector corresponding to the eigenvalue λ.

Theorem 4.1.1 *Let X be a linear space and $A : X \to X$ be a linear operator.*

(i) *If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of A with corresponding eigenvectors $x_1, \ldots, x_n$, then $\{x_1, \ldots, x_n\}$ is a linearly independent set.*

(ii) *If $R(A)$ is finite dimensional, then $\sigma_{\mathrm{eig}}(A)$ is a finite set.*

Proof. (i) This result is normally proved in a course in linear algebra, and hence its proof is left as an exercise.

(ii) Suppose $\sigma_{\text{eig}}(A)$ is an infinite set. Let (λ_n) be a sequence in $\sigma_{\text{eig}}(A)$ consisting of distinct nonzero terms, and for each $n \in \mathbb{N}$, let x_n be an eigenvector corresponding to the eigenvalue λ_n. By (i), $\{x_n : n \in \mathbb{N}\}$ is linearly independent. Since $Ax_n = \lambda_n x_n$ for all $n \in \mathbb{N}$, it follows that

$$\{x_n : n \in \mathbb{N}\} \subseteq R(A)$$

and hence, $R(A)$ is infinite dimensional. Thus, (ii) is proved. ∎

Remark 4.1.1 If X is finite dimensional, and if $[A]_E$ is the matrix representation of A with respect to a basis E of X, then $\sigma_{\text{eig}}(A)$ is the set of all eigenvalues of $[A]_E$. ♢

Example 4.1.1 Let $X = C[a,b]$ with $\|\cdot\|_\infty$ and let $u \in C[a,b]$. Let $A : X \to X$ be defined by

$$(Ax)(t) = u(t)x(t), \qquad t \in [a,b], \quad x \in X.$$

Clearly, $A \in \mathcal{B}(X)$. For $x \in C[a,b]$ and $\lambda \in \mathbb{K}$,

$$Ax = \lambda x \iff (u(t) - \lambda)x(t) = 0 \qquad \forall\, t \in [a,b].$$

Thus, $\lambda \in \sigma_{\text{eig}}(A)$ if and only if there exists an interval $I_\lambda \subseteq [a,b]$ such that $u(t) = \lambda$ for all $t \in I_\lambda$. In particular:

> If u is not a constant function on any subinterval of $[a,b]$, then $\sigma_{\text{eig}}(A) = \varnothing$.

□

Example 4.1.2 Let X be c_{00} or ℓ^p. Let (λ_n) be a sequence of scalars and $A : X \to X$ be defined by

$$(Ax)(j) = \lambda_j x(j), \quad j \in \mathbb{N}, \quad x \in X.$$

Then we have

$$Ae_n = \lambda_n e_n \quad \forall\, n \in \mathbb{N}.$$

Also, for $\lambda \in \mathbb{K}$ and for a nonzero $x \in X$,

$$Ax = \lambda x \iff \lambda \in \{\lambda_n : n \in \mathbb{N}\}.$$

Thus,

$$\sigma_{\text{eig}}(A) = \{\lambda_n : n \in \mathbb{N}\}.$$

□

In the last example, if $\lambda \notin \{\lambda_n : n \in \mathbb{N}\}$, but if λ is a limit point of $\{\lambda_n : n \in \mathbb{N}\}$, then there exists a subsequence of (λ_n), say (λ_{k_n}) which converges to λ, and in that case, taking the ℓ^p-norm, we have

$$\begin{aligned} \|Ae_{k_n} - \lambda e_{k_n}\|_p &\leq \|Ae_{k_n} - \lambda_{k_n} e_{k_n}\|_p + |\lambda_{k_n} - \lambda| \\ &= |\lambda_{k_n} - \lambda| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Definition 4.1.2 Let X be a normed linear space and $A : X \to X$ be a linear operator. A scalar λ is called an **approximate eigenvalue** of A if there exists (x_n) in X such that $\|x_n\| = 1$ for every $n \in \mathbb{N}$ and

$$\|Ax_n - \lambda x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

The set of all approximate eigenvalues of A is called the **approximate eigen spectrum** of A and it is denoted by $\sigma_{\text{app}}(A)$. ♢

Clearly,

$$\sigma_{\text{eig}}(A) \subseteq \sigma_{\text{app}}(A).$$

The proof of the following theorem is easy and hence it is left as an exercise.

Theorem 4.1.2 *If A is a linear operator on a normed linear space X and $\lambda \in \mathbb{K}$, then*

$$\lambda \in \sigma_{\text{app}}(A) \iff A - \lambda I \quad \textit{is not bounded below.}$$

Proof. Exercise. ∎

As a consequence of the above theorem, we have:

- $\lambda \in \sigma_{\text{app}}(A)$ implies $A - \lambda I$ does not have a bounded inverse.

Thus, if $\lambda \in \sigma_{\text{app}}(A)$, then even if the operator equation

$$Ax - \lambda x = y$$

has a unique solution for a given $y \in X$, the solution does not depend continuously on the data y.

To see this, suppose $\lambda \in \sigma_{\text{app}}(A)$ and $y \in R(A - \lambda I)$. Let $x \in X$ be such that $Ax - \lambda x = y$, and let (u_n) be such that $\|u_n\| = 1$ for every $n \in \mathbb{N}$ and $\|Au_n - \lambda u_n\| \to 0$. Then taking

$$v_n = Au_n - \lambda u_n, \quad y_n = y + v_n \quad \text{and} \quad x_n = x + u_n,$$

we have

$$Ax_n - \lambda x_n = y_n$$

for every $n \in \mathbb{N}$. Note that

$$\|y - y_n\| \to 0 \quad \text{but} \quad \|x - x_n\| = 1 \quad \forall\, n \in \mathbb{N}.$$

Example 4.1.3 Let X be c_{00} or ℓ^p with $\|\cdot\|_p$ and A be as in Example 4.1.2, i.e.,

$$(Ax)(j) = \lambda_j x(j), \quad j \in \mathbb{N}, \quad x \in X,$$

where (λ_n) be a sequence of scalars. Let $\Lambda := \{\lambda_n : n \in \mathbb{N}\}$. We show that

$$\sigma_{\rm app}(A) = {\rm cl}\,\Lambda.$$

We have already seen that ${\rm cl}\,\Lambda \subseteq \sigma_{\rm app}(A)$ (see the discussion preceding Definition 4.1.2). To see the reverse inclusion, let $\lambda \in \mathbb{K} \setminus {\rm cl}\,\Lambda$. Then, for every $x \in X$ and $j \in \mathbb{N}$, we have

$$|(Ax)(j) - \lambda x(j)| = |\lambda - \lambda_j|\,|x(j)| \geq d|x(j)|,$$

where $d := {\rm dist}\,(\lambda, \Lambda)$. Thus,

$$\|Ax - \lambda x\| \geq d\|x\| \quad \forall\, x \in X.$$

Consequently, $\lambda \notin \sigma_{\rm app}(A)$. □

In view of Example 4.1.3, we can state:

- Limit of a sequence of eigenvalues need not be an eigenvalue.

But, limit of a sequence of eigenvalues is always an approximate eigenvalue as the following theorem shows.

Theorem 4.1.3 *Let X be a normed linear space and $A : X \to X$ be a linear operator. Then*

$${\rm cl}\,\sigma_{\rm eig}(A) \subseteq \sigma_{\rm app}(A).$$

Proof. We have already observed that $\sigma_{\rm eig}(A) \subseteq \sigma_{\rm app}(A)$. Now, let (λ_n) is a sequence of eigenvalues of A which converges to λ. Let $x_n \in X$ be such that $\|x_n\| = 1$ and $Ax_n = \lambda_n x_n$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned} \|Ax_n - \lambda x_n\| &\leq \|Ax_n - \lambda_n x_n\| + |\lambda_n - \lambda| \\ &= |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, $\lambda \in \sigma_{\rm app}(A)$. ∎

In fact, the last theorem is a consequence of the following theorem as well.

Theorem 4.1.4 *Let A be a linear operator on a normed linear space X. Then $\sigma_{\rm app}(A)$ is a closed set.*

Proof. Let (λ_n) be a sequence in $\sigma_{\rm app}(A)$ such that $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{K}$. Note that, for every $x \in X$,

$$\begin{aligned} \|Ax - \lambda_n x\| &= \|(Ax - \lambda x) + (\lambda - \lambda_n)x\| \\ &\geq \|Ax - \lambda x\| - |\lambda - \lambda_n|\,\|x\| \end{aligned}$$

for every $n \in \mathbb{N}$. Assume for a moment that $\lambda \notin \sigma_{\rm app}(A)$. Then there exists $c > 0$ such that

$$\|Ax - \lambda x\| \geq c\|x\| \quad \forall\, x \in X.$$

Hence, for $k \in \mathbb{N}$ with $|\lambda - \lambda_k| < c/2$, we have

$$\|Ax - \lambda_k x\| \geq \frac{c}{2}\|x\| \quad \forall\, x \in X.$$

This is a contradiction to the fact that $\lambda_k \in \sigma_{\rm app}(A)$. ∎

Theorem 4.1.5 *If $A \in \mathcal{B}(X)$, then*

$$\sigma_{\rm app}(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \leq \|A\|\}.$$

Proof. Let $\lambda \in \sigma_{\rm app}(A)$. Let (x_n) in X be such that $\|x_n\| = 1$ for every $n \in \mathbb{N}$ and $\|Ax_n - \lambda x_n\| \to 0$. Then, for every $n \in \mathbb{N}$, we have

$$|\lambda| = \|\lambda x_n\| = \|Ax_n - (Ax_n - \lambda x_n)\| \leq \|A\| + \|Ax_n - \lambda x_n\|.$$

Thus, $|\lambda| \leq \|A\| + \|Ax_n - \lambda x_n\|$ for every $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain $|\lambda| \leq \|A\|$. ∎

Another proof. Let $\lambda \in \mathbb{K}$ be such that $|\lambda| > \|A\|$. Then, for every $x \in X$,

$$\|Ax - \lambda x\| \geq \|\lambda x\| - \|Ax\| \geq (|\lambda| - \|A\|)\|x\|.$$

Hence, $A - \lambda I$ is bounded below so that $\lambda \notin \sigma_{\rm app}(A)$. Therefore, if $\lambda \in \sigma_{\rm app}(A)$, then $|\lambda| \leq \|A\|$. ∎

Combining Theorems 4.1.4 and 4.1.5, we obtain the following.

Corollary 4.1.6 *If $A \in \mathcal{B}(X)$, then $\sigma_{\rm app}(A)$ is a compact set.*

Example 4.1.4 Consider the operator A in Example 4.1.1, i.e., $X = C[a,b]$ with $\|\cdot\|_\infty$ and $A : X \to X$ is defined by

$$(Ax)(t) = u(t)x(t), \qquad t \in [a,b], \quad x \in X,$$

where $u \in C[a,b]$. We have seen that $\sigma_{\rm eig}(A) = \varnothing$. Now, we show that

$$\sigma_{\rm app}(A) = {\rm cl}\,\{u(t) : t \in [a,b]\}.$$

For this, first let $\lambda \notin {\rm cl}\, S$, where $S = \{u(t) : t \in [a,b]\}$. Then, $d := \inf\{|\lambda - \mu| : \mu \in S\} > 0$ so that we obtain

$$\|Ax - \lambda x\|_\infty \geq d\|x\|_\infty \quad \forall\, x \in C[a,b].$$

Hence, $\sigma_{\rm app}(A) \subseteq {\rm cl}\, S$.

To obtain the reverse inclusion, by the closedness of $\sigma_{\rm app}(A)$, it is enough to prove $S \subseteq \sigma_{\rm app}(A)$. So, let $\lambda \in S$ and $t_0 \in [a,b]$ be such that $u(t_0) = \lambda$. For each $n \in \mathbb{N}$, let I_n be an open interval containing λ such that its length is less than $1/n$. Since $u \in C[a,b]$, there exists an interval $J_n \subseteq [a,b]$ containing t_0 such that $u(J_n) \subseteq I_n$. Let $x_n \in C[a,b]$ be such that $\|x_n\|_\infty = 1$ and $x_n(t) = 0$ for $t \notin J_n$. Then we have

$$\|Ax_n - \lambda x_n\|_\infty = \sup_{t \in J_n} |u(t) - \lambda|\,|x_n(t)| \leq \sup_{t \in J_n} |u(t) - \lambda| \leq \frac{1}{n} \to 0.$$

Thus, $S \subseteq \sigma_{\rm app}(A)$. Thus, we have shown that $\sigma_{\rm app}(A) = {\rm cl}\, S$. □

Next we show some nice properties of the approximate eigenspectra of compact operators. Before that, we prove the following lemma which will be used subsequently.

Lemma 4.1.7 (Riesz lemma) *Let X be a normed linear space, X_0 be a proper closed subspace of X and $0 < r < 1$. Then there exists $x_r \in X$ such that*

$$\|x_r\| = 1 \quad \textit{and} \quad {\rm dist}\,(x_r, X_0) \geq r.$$

Proof. Since X_0 is a proper closed subspace of X, there exists $x \in X \setminus X_0$ such that $d := {\rm dist}\,(x, X_0) > 0$. Since $d/r > d$, there exists $u \in X_0$ such that $\|u - x\| \leq d/r$. Let

$$x_r = \frac{x - u}{\|x - u\|}.$$

Then $\|x_r\| = 1$ and

$$\operatorname{dist}(x_r, X_0) = \frac{\operatorname{dist}(x, X_0)}{\|x - u\|} \geq \frac{d}{d/r} = r.$$

This completes the proof. ∎

Theorem 4.1.8 *Let $A \in \mathcal{K}(X)$. Then the following hold.*

(i) $\sigma_{\text{app}}(A) \setminus \{0\} = \sigma_{\text{eig}}(A) \setminus \{0\}$,

(ii) *$N(A - \lambda I)$ is finite dimensional for every nonzero $\lambda \in \sigma_{\text{eig}}(A)$.*

(iii) *0 is the only possible limit point of $\sigma_{\text{eig}}(A)$, and $\sigma_{\text{eig}}(A)$ is a countable set.*

Proof. (i) Its enough to prove that $\sigma_{\text{app}}(A) \setminus \{0\} \subseteq \sigma_{\text{eig}}(A) \setminus \{0\}$. So, let $\lambda \in \sigma_{\text{app}}(A) \setminus \{0\}$ and let (x_n) in X such that $\|x_n\| = 1$ for every $n \in \mathbb{N}$ and $\|Ax_n - \lambda x_n\| \to 0$ as $n \to \infty$. Since A is a compact operator, there exists a subsequence (x_{k_n}) such that $Ax_{k_n} \to y$ for some $y \in X$. Hence,

$$x_{k_n} = \frac{1}{\lambda}[Ax_{k_n} - (Ax_{k_n} - \lambda x_{k_n})] \to \frac{y}{\lambda}$$

so that we have

$$Ax_{k_n} \to \frac{Ay}{\lambda}.$$

Since $Ax_{k_n} \to y$, we obtain $Ay = \lambda y$. Also, $y \neq 0$, since $\|x_{k_n}\| = 1$ and $\lambda \neq 0$. Thus, we have proved that $\lambda \in \sigma_{\text{eig}}(A)$.

(ii) Let λ be a nonzero eigenvalue of A. Suppose $N(A - \lambda I)$ is infinite dimensional. Let $\{x_n : n \in \mathbb{N}\}$ be a linearly independent subset of $N(A - \lambda I)$. Let $X_0 = \{0\}$ and for $n \in \mathbb{N}$, let

$$X_n := \operatorname{span}\{x_1, \ldots, x_n\}.$$

Since $\{x_n : n \in \mathbb{N}\}$ is linearly independent, by Riesz lemma (Lemma 4.1.7), for each $n \in \mathbb{N}$, there exists $u_n \in X_n$ such that

$$\|u_n\| = 1 \quad \text{and} \quad \operatorname{dist}(u_n, X_{n-1}) \geq \frac{1}{2}.$$

Note that for $n, m \in \mathbb{N}$ with $n > m$, $u_m \in X_{n-1}$ and

$$\|Au_n - Au_m\| = \|\lambda(u_n - u_m)\| \geq \frac{|\lambda|}{2}.$$

Thus, (Au_n) does not have a Cauchy subsequence so that (Au_n) does not have a convergent subsequence.

(iii) For $r > 0$, let

$$\Delta_r := \{\lambda \in \sigma_{\text{eig}}(A) : |\lambda| \geq r\}.$$

It is enough to prove that prove that Δ_r is a finite set (*Why?*). Assume for a moment that Δ_r is an infinite set for some $r > 0$. Let (λ_n) be a sequence of distinct elements from Δ_r. For each $n \in \mathbb{N}$, let x_n be an eigen vector of A corresponding to the eigenvalue λ_n. Let $X_0 = \{0\}$ and for $n \in \mathbb{N}$, let

$$X_n := \text{span}\,\{x_1, \ldots, x_n\}.$$

Since $\{x_n : n \in \mathbb{N}\}$ is linearly independent, by Riesz lemma (Lemma 4.1.7), for each $n \in \mathbb{N}$, there exists $u_n \in X_n$ such that

$$\|u_n\| = 1 \quad \text{and} \quad \text{dist}\,(u_n, X_{n-1}) \geq \frac{1}{2}.$$

Note that for $n, m \in \mathbb{N}$ with $n > m$,

$$Au_n - Au_m = \lambda_n u_n + (Au_n - \lambda_n u_n) - Au_m,$$

where

$$Au_n - \lambda_n u_n \in X_{n-1} \quad \text{and} \quad Au_m \in X_{n-1}.$$

Hence,

$$\|Au_n - Au_m\| \geq \text{dist}\,(\lambda_n u_n, X_{n-1}) = |\lambda_n| \text{dist}\,(u_n, X_{n-1}) \geq \frac{r}{2}.$$

Thus, (Au_n) does not have a Cauchy subsequence so that (Au_n) does not have a convergent subsequence. ∎

By the very nature of a compact operator, it is clear that if X is infinite dimensional and $A \in \mathcal{K}(X)$, then

$$0 \in \sigma_{\text{app}}(S).$$

In view of Theorem 4.1.8, one may enquire whether every compact operator has an eigenvalue. The answer is, in general, not in affirmative as the following example shows.

Example 4.1.5 Let $X = L^2[a,b]$ and

$$(Ax)(s) = \int_a^s x(t)\,dt, \quad x \in L^2[a,b].$$

Recall from Example 2.4.4 that A is a compact operator.

Now, let $\lambda \in \mathbb{K}$ and $x \in L^2[a,b]$ be such that $Ax = \lambda x$. Then we have

$$\lambda x(s) = \int_a^s x(t)\,dt, \quad \text{for almost all}\, s \in [a,b].$$

Recall from *fundamental theorem of Lebesgue integration*, that for $u \in L^2[a,b]$, if

$$v(s) = \int_a^s u(t)\,dt, \quad s \in [a,b],$$

then v is *absolutely continuous*, and $v' = u$ almost everywhere. Thus, if $\lambda = 0$, then $x = 0$, and if $\lambda \neq 0$, then x is absolutely continuous, $x(a) = 0$ and

$$x'(s) = \frac{x(s)}{\lambda} \quad \text{a.e.},$$

so that, in this case also, we obtain $x = 0$. Thus, A does not have any eigenvalue. □

4.2 Resolvent Set and Spectrum

Throughout this chapter, we assume that X is a normed linear space and $A : X \to X$ is a linear operator. For deriving interesting and important results, we shall assume further properties on X and A.

Definition 4.2.1 The set of all $\lambda \in \mathbb{K}$ such that $A - \lambda I$ is bijective and $(A - \lambda I)^{-1}$ is continuous is called the **resolvent set** of A, and it is denoted by $\rho(A)$. The compliment of $\rho(A)$ is called the **spectrum** of A, and it is denoted by $\sigma(A)$. ◊

Thus, for $\lambda \in \mathbb{K}$,

$$\begin{aligned} \lambda \in \rho(A) \quad &\Longleftrightarrow \quad A - \lambda I \text{ is bijective and } (A - \lambda I)^{-1} \in \mathcal{B}(X), \\ \lambda \in \sigma(A) \quad &\Longleftrightarrow \quad \lambda \notin \rho(A). \end{aligned}$$

The following theorem is an immediate consequence of bounded inverse theorem (Corollary 3.1.8).

Theorem 4.2.1 *If X is a Banach space and $A \in \mathcal{B}(X)$, then*

$$\sigma(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not bijective } \}.$$

Theorem 4.2.2 *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then*

$$\sigma(A) = \sigma_{app}(A) \cup \sigma_{\rm com}(A),$$

where $\sigma_{\rm com}(A)$, called the **compression spectrum** *of A, is the set all those $\lambda \in \mathbb{K}$ such that $R(A - \lambda I)$ not dense in X.*

Proof. Clearly,

$$\sigma_{app}(A) \cup \sigma_{\rm com}(A) \subseteq \sigma(A).$$

Next, let $\lambda \notin \sigma_{app}(A) \cup \sigma_{\rm com}(A)$. Then $A - \lambda I$ is bounded below and $R(A - \lambda I)$ is dense. Hence, $A - \lambda I$ is bijective so that by Theorem 4.2.1,

$$\sigma(A) \subseteq \sigma_{app}(A) \cup \sigma_{\rm com}(A).$$

This completes the proof. ∎

Clearly

$$\sigma_{\rm eig}(A) \subseteq \sigma_{\rm app}(A) \subseteq \sigma(A).$$

We have already seen the case where the first inclusion above is strict. The following example shows that the second inclusion also can be strict.

Example 4.2.1 Let $X = c_{00}$ with ℓ^p-norm and A be the *right shift operator*, that is,

$$(Ax)(i) = \begin{cases} 0, & i = 1, \\ x(i-1), & i \neq 1. \end{cases}$$

Then $\|Ax\| \geq \|x\|$ for all $x \in X$ and $e_1 \notin R(A)$. Thus,

$$0 \in \sigma(A) \setminus \sigma_{\rm app}(A).$$

Also, for any $\lambda \in \mathbb{K}$ with $|\lambda| < 1$, we have

$$\|Ax - \lambda x\| \geq \|Ax\| - \|\lambda x\| \geq (1 - |\lambda|)\|x\| \quad \forall\, x \in X,$$

and we see that $e_1 \notin R(A - \lambda I)$ so that

$$\{\lambda \in \mathbb{K} : |\lambda| < 1\} \subseteq \sigma(A) \setminus \sigma_{\rm app}(A).$$

We shall see in Example 4.2.3 that, for this operator A,

$$\sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\} \quad \text{and} \quad \sigma_{\text{app}}(A) = \{\lambda \in \mathbb{K} : |\lambda| = 1\}.$$

□

Example 4.2.2 Let X be c_{00} or ℓ^p with $\|\cdot\|_p$ and A be as in Example 4.1.3, i.e.,

$$(Ax)(j) = \lambda_j x(j), \quad j \in \mathbb{N}, \quad x \in X,$$

where (λ_n) be a sequence of scalars. We have seen that

$$\sigma_{\text{eig}}(A) = \Lambda \quad \text{and} \quad \sigma_{\text{app}}(A) = \text{cl}\,\Lambda,$$

where $\Lambda := \{\lambda_n : n \in \mathbb{N}\}$. Thus, $\text{cl}\,\Lambda \subseteq \sigma(A)$. Now, we show that

$$\sigma(A) = \text{cl}\,\Lambda.$$

Let $\lambda \notin \text{cl}\,\Lambda$. We know that $A - \lambda I$ is one-one. For $y \in X$, let $x \in X$ be defined by

$$x(j) = \frac{y(j)}{\lambda_j - \lambda} \qquad \forall j \in \mathbb{N}.$$

Since $|\lambda_j - \lambda| \ge d := \text{dist}\,(\lambda, \text{cl}\,\Lambda)$, $x \in X$, and $Ax - \lambda x = y$ so that $A - \lambda I$ is onto as well. Further,

$$\|(A - \lambda I)^{-1} y\| = \|x\| \le \frac{\|y\|}{d}$$

so that $(A - \lambda I)^{-1}$ is a bounded operator. Thus, $\sigma(A) \subseteq \text{cl}\,\Lambda$, and we have completed the proof of $\sigma(A) = \text{cl}\,\Lambda$. □

We have seen that $\sigma_{\text{app}}(A)$ is a closed set, and if $A \in \mathcal{B}(X)$, then $\sigma_{\text{app}}(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \le \|A\|\}$. Now, we show that these results hold for $\sigma(A)$ whenever X is a Banach space.

Theorem 4.2.3 *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then the following hold.*

(i) *$\sigma(A)$ is a bounded set. More precisely,*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \le \|A\|\}.$$

(ii) *$\rho(A)$ is an open set. More precisely, for each $\lambda_0 \in \rho(A)$,*

$$\{\lambda \in \mathbb{K} : |\lambda - \lambda_0| < 1/\|(A - \lambda_0 I)^{-1}\|\} \subseteq \rho(A).$$

In particular, $\sigma(A)$ is a compact subset of $\mathbb{K}$.

Proof. (i) Let $\in \mathbb{K}$ be such that $|\lambda| > \|A\|$. Then, for every $x \in X$,

$$\|(A - \lambda I)x\| \geq \|\lambda x\| - \|Ax\| \geq (|\lambda| - \|A\|)\|x\|.$$

From this, it follows that $A - \lambda I$ is one–one, $R(A - \lambda I)$ is closed and $A - \lambda I$ has a continuous inverse from its range. Hence, to complete the proof of (i), using Theorem 4.2.1, it is enough to prove that $R(A - \lambda I) = X$. Suppose this is not true. Then, by a consequence of Hahn Banach theorem (see Corollary 3.2.3), there exists $f \in X'$ such that $\|f\| = 1$ and $f(y) = 0$ for all $y \in R(A - \lambda I)$. In particular, $f(Ax - \lambda x) = 0$ for all $x \in X$. Hence,

$$|\lambda|\,\|x\| = \|\lambda x\| = \|f(Ax)\| \leq \|f\|\,\|A\|\,\|x\| = \|A\|\,\|x\| \qquad \forall\, x \in X.$$

Thus,

$$(|\lambda| - \|A\|)\|x\| = 0 \qquad \forall\, x \in X.$$

This is a contradiction, since $|\lambda| > \|A\|$. Thus, $|\lambda| > \|A\|$ implies $A - \lambda I$ is bijective.

(ii) Let $\lambda_0 \in \rho(A)$ and $\lambda \in \mathbb{K}$ be such that

$$|\lambda - \lambda_0| < 1/\|(A - \lambda_0 I)^{-1}\|.$$

Since,

$$\begin{aligned} A - \lambda I &= (A - \lambda_0 I) - (\lambda - \lambda_0)I \\ &= [I - (\lambda - \lambda_0)(A - \lambda_0 I)^{-1}](A - \lambda_0 I), \end{aligned}$$

by (i) and Theorem 4.2.1, $\lambda \in \rho(A)$. Thus,

$$\{\lambda \in \mathbb{K} : |\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}\} \subseteq \rho(A).$$

Hence, $\rho(A)$ is an open set and consequently, $\sigma(A)$ is a closed set.

By (i) and (ii), $\sigma(A)$ is a compact subset of $\mathbb{K}$. ∎

Corollary 4.2.4 *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then*

$$\sigma(A) \subseteq \bigcap_{n=1}^{\infty} \{\lambda \in \mathbb{K} : |\lambda| \leq \|A^n\|^{1/n}\}.$$

Proof. It is enough to show that

$$\bigcup_{n=1}^{\infty}\{\lambda \in \mathbb{K} : |\lambda|^n > \|A^n\|\} \subseteq \rho(A).$$

So, let $\lambda \in \mathbb{K}$ is such that $|\lambda|^n > \|A^n\|$ for some $n \in \mathbb{N}$. Note that

$$A^n - \lambda^n I = (A - \lambda I)\sum_{j=1}^{n}\lambda^{j-1}A^{n-j} = \Big[\sum_{j=1}^{n}\lambda^{j-1}A^{n-j}\Big](A - \lambda I)$$

By Theorem 4.2.3(i), $A^n - \lambda^n I$ is bijective. Hence, from the above equalities, $A - \lambda I$ is bijective. Since X is a Banach space, by Theorem 4.2.1, $\lambda \in \rho(A)$. ∎

Definition 4.2.2 Let $A \in \mathcal{B}(X)$. Then the number

$$r_\sigma(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

is called the **spectral radius** of A. ◊

By Corollary 4.2.4, if X is a Banach space and $A \in \mathcal{B}(X)$, then

$$r_\sigma(A) \le \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}.$$

In fact, we have the following theorem. We omit its proof. Interested reader may see the proof in [5].

Theorem 4.2.5 (Gelfand–Mazur Theorem) *Let X be a Banach space over the complex field $\mathbb{C}$ and $A \in \mathcal{B}(X)$. Then*

(i) **(Gelfand–Mazur Theorem)** *$\sigma(A)$ is nonempty,*

(ii) **(Spectral radius formula)** $\lim_{n\to\infty} \|A^n\|^{1/n}$ *exists and*

$$r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Theorem 4.2.6 *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then, every boundary point of $\sigma(A)$ is an approximate eigenvalue of A.*

Proof. Let λ be a boundary point of $\sigma(A)$. Then $\lambda \in \sigma(A)$ and there exists a sequence (μ_n) in $\rho(A)$ such that $\mu_n \to \lambda$. Suppose $\lambda \notin \sigma_{\rm app}(A)$, and let $c > 0$ be such that

$$\|Ax - \lambda x\| \geq c\|x\| \quad \forall\, x \in X.$$

Let $N \in \mathbb{N}$ be such that $|\lambda - \mu_N| < c/2$. Then, we have

$$\begin{aligned} \|Ax - \mu_N x\| &= \|(Ax - \lambda x) - (\mu_N - \lambda)x\| \\ &\geq \|Ax - \lambda x\| - |\mu_N - \lambda|\|x\| \\ &\geq (c - |\mu_N - \lambda|)\|x\| \\ &> \frac{c}{2}\|x\| \end{aligned}$$

Hence,

$$\|(A - \mu_N I)^{-1}\| < \frac{2}{c}$$

so that

$$\|(\lambda - \mu_N)(A - \mu_N I)^{-1}\| < 1.$$

Therefore, by Theorem 4.2.3(ii), $\lambda \in \rho(A)$. This is a contradiction to the fact that $\lambda \in \sigma(A)$. ∎

Example 4.2.3 Consider the Example 4.2.1. We show that

$$\sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\} \quad \text{and} \quad \sigma_{\rm app}(A) = \{\lambda \in \mathbb{K} : |\lambda| = 1\}.$$

We have seen that

$$\{\lambda \in \mathbb{K} : |\lambda| < 1\} \subseteq \sigma(A) \setminus \sigma_{\rm app}(A).$$

It can also be seen that $\|A\| \leq 1$. Thus,

$$\{\lambda \in \mathbb{K} : |\lambda| < 1\} \subseteq \sigma(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}.$$

Hence, by the closedness of $\sigma(A)$, $\sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$. The above observations together with Theorem 4.2.6 imply that $\sigma_{\rm app}(A) = \{\lambda \in \mathbb{K} : |\lambda| = 1\}$. □

Example 4.2.4 Let $X = C[a, b]$ with $\|\cdot\|_\infty$ and let $u \in C[a, b]$. Let $A : X \to X$ be defined by

$$(Ax)(t) = u(t)x(t), \qquad t \in [a, b], \quad x \in X.$$

Clearly, $A \in \mathcal{B}(X)$. We show that

$$\sigma(A) = \mathrm{cl}\, S,$$

where $S := \{u(t) : t \in [a, b]\}$. Recall from Example 4.1.4 that $\sigma_{\mathrm{app}}(A) = \mathrm{cl}\, S$. Hence, it is enough to show that $\sigma(A) \subseteq \mathrm{cl}\, S$.

Suppose $\lambda \notin \mathrm{cl}\, S$. Then $A - \lambda I$ is one-one. Also, for every $y \in C[a, b]$, the function $x \in C[a, b]$ defined by

$$x(t) = \frac{y(t)}{u(t) - \lambda}, \quad t \in [a, b],$$

satisfies the equation $Ax - \lambda x = y$. Thus, for all $\lambda \notin \mathrm{cl}\, S$, $A - \lambda I$ is bijective. Since X is a Banach space, by Theorem 4.2.1, $\sigma(A) \subseteq S$. Thus, we have proved that $\sigma(A) = \mathrm{cl}\, S$. □

4.3 Spectral Results for Self Adjoint, Normal and Unitary Operators

We know from linear algebra that if A is a self adjoint operator on a finite dimensional inner product space, then its eigen spectrum is nonempty finite set of real numbers, irrespective of whether the scalar field is $\mathbb{R}$ or $\mathbb{C}$. One may wonder whether the same can be said about the spectrum of a self adjoint operator on a (possibly infinite dimensional) Hilbert space. Yes, we can. We shall move towards the justification of this claim.

Throughout this section, we consider X to be a Hilbert space and $A \in \mathcal{B}(X)$. Recall that A is

- *self-adjoint* if $A^* = A$,
- *normal* if $A^*A = AA^*$, and
- *unitary* if $A^*A = I = AA^*$.

We shall make use of the following easily verifiable result.

Lemma 4.3.1 *Let $A \in \mathcal{B}(X)$. Then*

$$R(A)^\perp = N(A^*).$$

Theorem 4.3.2 *Let $A \in \mathcal{B}(X)$ and $\lambda \in \mathbb{K}$. Then*

$R(A - \lambda I)$ is dense in X if and only if $\bar{\lambda} \notin \sigma_{\mathrm{eig}}(A^*)$.

Proof. We note that for $\in \mathbb{K}$, $(A - \lambda I)^* = A^* - \bar{\lambda} I$. Hence, by Lemma 4.3.1, replacing A by $A - \lambda I$, we obtain

$$\begin{aligned} \bar{\lambda} \notin \sigma_{\rm eig}(A^*) &\iff N(A^* - \bar{\lambda} I) = \{0\} \\ &\iff R(A - \lambda I)^\perp = \{0\} \\ &\iff R(A - \lambda I) \text{ dense in } X. \end{aligned}$$

The last equivalence is a consequence of projection theorem. This competes the proof. ∎

In view of the aove theorem together with Theorem 4.2.2, we have the following corollary.

Corollary 4.3.3 *Let $A \in \mathcal{B}(X)$. Then*

$$\sigma(A) = \sigma_{\rm app}(A) \cup \{\lambda \in \mathbb{K} : \bar{\lambda} \in \sigma_{\rm eig}(A^*)\}.$$

Clearly, if A is self adjoint, then

$$\sigma_{\rm eig}(A) \subseteq \mathbb{R}.$$

We, in fact, have the following.

Theorem 4.3.4 *Let A be a self-adjoint operator. Then*

$$\sigma(A) \subseteq \mathbb{R}.$$

Proof. If $\mathbb{K} = \mathbb{R}$, then there is nothing to prove. Hence, assume that $\mathbb{K} = \mathbb{C}$. Let $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. It is enough to show that $\lambda \in \rho(A)$. For this first we note that, for every $x \in X$,

$$\begin{aligned} \|Ax - \lambda x\|^2 &= \langle (A - \alpha I)x + i\beta x, (A - \alpha I)x + i\beta x \rangle \\ &= \|(A - \alpha I)x\|^2 + |\beta|^2 \|x\|^2. \end{aligned}$$

To obtain the above, we used the fact that

$$\langle (A - \alpha I)x, \beta x \rangle = \langle \beta x, (A - \alpha I)x \rangle.$$

Thus, $A - \lambda I$ is bounded below, so that it is one-one and $R(A - \lambda I)$ is closed. Similarly, $A - \bar{\lambda} I$ is also one-one. Hence, by Lemma 4.3.1,

$$[R(A - \lambda I)]^\perp = N(A^* - \bar{\lambda} I) = N(A - \bar{\lambda} I) = \{0\}.$$

Consequently, $R(A - \lambda I)$ is dense in X, so that by the closedness of $R(A - \lambda I)$, $A - \lambda I$ is onto. ∎

For normal operators we have the following.

Theorem 4.3.5 *Let A be a normal operator and $\lambda \in \mathbb{K}$. Then*

(i) *For $x \in X$, $Ax = \lambda x \iff A^*x = \bar{\lambda}x$. In particular,*

$$\lambda \in \sigma_{\rm eig}(A) \iff \bar{\lambda} \in \sigma_{\rm eig}(A^*).$$

(ii) $\lambda, \mu \in \mathbb{K}$, $\lambda \neq \mu \Longrightarrow N(A - \lambda I) \perp N(A - \mu I)$.

(iii) $\sigma(A) = \sigma_{\rm app}(A)$.

Proof. Let $x \in X$ and $\lambda \in \mathbb{K}$. Then, using the fact that A is normal, we have

$$\begin{aligned} \|Ax - \lambda x\|^2 &= \langle (A-\lambda I)x,\, (A-\lambda I)x\rangle \\ &= \langle x,\, (A^* - \bar{\lambda} I)(A - \lambda I)x\rangle \\ &= \langle x,\, (A - \lambda I)(A^* - \bar{\lambda} I)x\rangle \\ &= \langle (A^* - \bar{\lambda} I)x,\, (A^* - \bar{\lambda} I)x\rangle \\ &= \|A^*x - \bar{\lambda}x\|^2. \end{aligned}$$

From this, (i) follows.

Now, let $\lambda, \mu \in \mathbb{K}$ such that $\lambda \neq \mu$. Let $x \in N(A - \lambda I)$ and $y \in N(A - \mu I)$. Then, using (i), we have

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Ax, y\rangle = \langle x, A^*y\rangle = \langle x, \bar{\mu}y\rangle = \mu\langle x, y\rangle.$$

Thus, $\lambda \neq \mu$ implies $\langle x, y\rangle = 0$. Thus, (ii) is proved.

Now, (i) and Corollary 4.3.3 imply (iii). ∎

Next result is concerned about the spectra of unitary operators.

Theorem 4.3.6 *Let A be a unitary operator and $X \neq \{0\}$. Then*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| = 1\}.$$

Further, if $\sigma(A) \neq \varnothing$, then

$$r_\sigma(A) = 1.$$

Proof. Since $A^*A = I = AA^*$,

$$\|Ax\| = \|x\| = \|A^*x\| \qquad \forall\, x \in X.$$

Hence, $\|A\| = 1$. Now, let $\lambda \in \mathbb{K}$ be such that $|\lambda| \neq 1$. Then for every $x \in X$, we have

$$\|Ax - \lambda x\| \geq |\, \|Ax\| - |\lambda|\, \|x\|\, | = |\, 1 - |\lambda|\, |\, \|x\|.$$

Consequently, by Theorem 4.3.5 (iii), $\sigma(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| = 1\}$. The last part follows from the definition of $r_\sigma(A)$. ∎

Definition 4.3.1 For $A \in \mathcal{B}(X)$, the set

$$W(A) := \{\langle Ax, x\rangle : \|x\| = 1\}$$

is called the **numerical range** of A, and

$$r_W(A) := \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}$$

is called the **numerical radius** of A. ◊

Observe:

- A self adjoint $\Longrightarrow W(A) \subseteq \mathbb{R}$.

Converse of the above need not be true. For instance, if $\mathbb{K} = \mathbb{R}$, then $W(A) \subseteq \mathbb{R}$ even if A is not self adjoint. However, the converse holds if the scalar field is $\mathbb{C}$ (see [5]).

Definition 4.3.2 If $W(A) \subseteq [0, \infty)$, then A is called a **positive operator** . ◊

Notation 4.3.1 For $A \in \mathcal{B}(X)$ with $W(A) \subseteq \mathbb{R}$, let us use the following notations:

$$\begin{aligned} \alpha_A &:= \inf\{\langle Ax, x\rangle : \|x\| = 1\}, \\ \beta_A &:= \sup\{\langle Ax, x\rangle : \|x\| = 1\}. \end{aligned}$$

◊

Theorem 4.3.7 *Let $A \in \mathcal{B}(X)$ be self adjoint. Then*

$$\|A\| = r_W(A) = \max\{|\alpha_A|, |\beta_A|\},$$

and if A is positive self adjoint, then $\|A\| = \beta_A$.

Proof. Follows from Theorem 2.2.5. ∎

Lemma 4.3.8 *Suppose A is a positive self-adjoint operator. Then*

$$\beta_A \in \sigma(A).$$

In particular, if A is positive self adjoint, then $r_\sigma(A) = \|A\|$.

Proof. Let (x_n) in X be such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\langle Ax_n, x_n\rangle \to \beta_A$ as $n \to \infty$. Note that

$$\begin{aligned} \|Ax_n - \beta_A x_n\|^2 &= \|Ax_n\|^2 - 2\beta_A\langle Ax_n, x_n\rangle + \beta_A^2 \\ &\leq \|A\|^2 - 2\beta_A\langle Ax_n, x_n\rangle + \beta_A^2. \end{aligned}$$

Since $\langle Ax_n, x_n\rangle \to \beta_A$ as $n \to \infty$ and $\beta_A = \|A\|$ (see Theorem 4.3.7), it follows from the above inequality that $\|Ax_n - \beta_A x_n\| \to 0$ as $n \to \infty$. Thus, $\beta_A \in \sigma_{\text{app}}(A) = \sigma(A)$. ∎

Theorem 4.3.9 *Suppose A is a self-adjoint operator. Then*

$$r_\sigma(A) = \|A\|.$$

In particular, there exists $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$.

Proof. In view of Theorem 4.3.7, it is enough to prove that

$$\{\alpha_A, \beta_A\} \subseteq \sigma(A).$$

For this purpose, we may first observe that

$$B := A - \alpha_A I \quad \text{and} \quad C := \beta_A I - A$$

are positive self adjoint operators. Therefore, by what we have proved in the previous paragraph,

$$\beta_B \in \sigma(B), \quad \beta_C \in \sigma(C).$$

But,

$$\begin{gathered} \beta_B = \sup\{\langle (A - \alpha_A I)x, x\rangle : \|x\| = 1\} = \beta_A - \alpha_A, \\ \beta_C = \sup\{\langle (\beta_A I - A)x, x\rangle : \|x\| = 1\} = \beta_A - \alpha_A, \\ \sigma(B) = \{\lambda - \alpha_A : \lambda \in \sigma(A)\}, \\ \sigma(C) = \{\beta_A - \lambda : \lambda \in \sigma(A)\}. \end{gathered}$$

Hence, there there exists $\lambda, \mu \in \sigma(A)$ such that

$$\beta_A - \alpha_A = \lambda - \alpha_A \quad \beta_A - \alpha_A = \beta_A - \mu.$$

Consequently, $\beta_A = \lambda \in \sigma(A)$ and $\alpha_A = \mu \in \sigma(A)$. This completes the proof of the theorem. ∎

Corollary 4.3.10 *If $A \in \mathcal{B}(X)$ is a compact self adjoint operator, then there exists $\lambda \in \sigma_{\rm eig}(A)$ such that $|\lambda| = \|A\|$.*

Remark 4.3.1 Theorem 4.3.9, in particular, shows that if A is a self adjoint operator, then the fact that $\sigma(A) \neq \varnothing$ (cf. Theorem 4.2.5) holds for a real Hilbert space as well. ◊

By Theorem 4.3.9, if A is a self adjoint operator, then $\sigma(A) \neq \varnothing$. However, the eigenspectrum can be empty even if A is self adjont as the following example shows.

Example 4.3.1 Let $X = L^2[a,b]$ and

$$(Ax)(t) = tx(t) \text{ for almost all } t \in [a,b].$$

Note that A is a self adjoint operator.

Now, for $\lambda \in \mathbb{K}$ and $x \in L^2[a,b]$,

$$\begin{aligned} Ax = \lambda x \quad &\Longleftrightarrow \quad (\lambda - t)x(t) = 0 \text{ for almost all } t \in [a,b] \\ &\Longleftrightarrow \quad x = 0. \end{aligned}$$

Thus, A does not have any eigenvalue. □

Recall from Theorem 4.3.5 (iii) that if A is a normal operator, then $\sigma(A) = \sigma_{\rm app}(A)$. For a general bounded operator, we have the following result.

Theorem 4.3.11 *For $A \in \mathcal{B}(X)$,*

$$\sigma(A) = \sigma_{\rm app}(A) \cup \{\lambda \in \mathbb{K} : \bar{\lambda} \in \sigma_{\rm eig}(A^*).$$

Proof. By Theorem 4.3.2, for $\lambda \in \mathbb{K}$,

$$R(A - \lambda I) \text{ not dense in } X \iff \bar{\lambda} \in \sigma_{\rm eig}(A^*).$$

Hence, the result is a consequence of Theorem 4.2.2. ∎

4.4 Spectral Representations

Recall from linear algebra that if X is a finite dimensional inner product space and $A : X \to X$ is a self adjoint operator, then A can be represented as

$$Ax = \sum_{j=1}^{k} \lambda_j \langle x, u_j \rangle u_j, \qquad x \in X,$$

where $\lambda_1, \ldots, \lambda_k$ are nonzero real numbers and $\{u_1, \ldots, u_k\}$ is an orthonormal set in X.

In this section we prove that an analogous representation is possible if A is a compact self adjoint operator in a general Hilbert space. Our proof includes the case of finite dimensional case as well.

First, let us recall the following facts about a compact operator A on a general Banach space (cf. Theorem 4.1.8):

1. Eigen spectrum of A is countable,
2. 0 is the only possible limit point of he eigen spectrum of A, and
3. Eigen space associated with every nonzero eigenvalue is finite dimensional.
4. Every nonzero approximate eigenvalue of A is an eigenvalue.

Also for a self adjoint operator A on a Hilbert space A, we know the following (cf. Theorems 4.3.5 and Corollary 4.3.10):

1. Eigen vectors corresponding to distinct eigenvalues of A are orthogonal.
2. A has an eigenvalue λ such that $|\lambda| = \|A\|$.

We shall also make use of a few simple-minded lemmas.

Lemma 4.4.1 *Let A be a self adjoint operator on a Hilbert space X and X_0 be a closed subspace of X. Then*

$$A(X_0) \subseteq X_0 \iff A(X_0^\perp) \subseteq X_0^\perp.$$

Proof. Suppose $A(X_0) \subseteq X_0$. Let $x \in X_0^\perp$. Then for every $y \in X_0$, $Ay \in X_0$ so that

$$\langle Ax, y\rangle = \langle x, Ay\rangle = 0.$$

Thus, $A(X_0^\perp) \subseteq X_0^\perp$. Also, by projection theorem, $X_0^{\perp\perp} = X_0$ so that from what we have proved,

$$A(X_0^\perp) \subseteq X_0^\perp \Longrightarrow A(X_0) = A(X_0^{\perp\perp}) \subseteq X_0^{\perp\perp} = X_0.$$

This completes the proof. ∎

Definition 4.4.1 Let A be a linear operator on a linear space X. A subspace X_0 of X is said to be **invariant** under A or an **invariant subspace** for A if $A(X_0) \subseteq X_0$. ◊

Example 4.4.1 Let A be a linear operator on a linear space X and let $\{\lambda_1, \ldots, \lambda_k\} \subseteq \mathbb{K}$. Then it can be easily seen that

$$X_0 = N(A - \lambda_1 I) + \ldots + N(A - \lambda_k I)$$

is invariant under A. □

Suppose A is a self adjoint operator on a Hilbert space X and X_0 is an invariant subspace for X. Then, by Lemma 4.4.1, $X_0^\perp$ is also invariant under A. Hence, it can be seen that

$$A_1 := A_{|X_0} \quad \text{and} \quad A_2 := A_{|X_0^\perp}$$

are self adjoint operators on X_0 and $X_0^\perp$, respectively.

Lemma 4.4.2 *Let A be a self adjoint operator on a Hilbert space X and X_0 be an invariant subspace for X. Let $A_1 := A_{|X_0}$ and $A_2 := A_{|X_0^\perp}$. Then*

$$\sigma_{\text{eig}}(A) = \sigma_{\text{eig}}(A_1) \cup \sigma_{\text{eig}}(A_2).$$

Proof. We observe that if $x \in X$ and $(u, v) \in X_0 \times X_0^\perp$ is such that $x = u + v$, then $x \neq 0$ if and only if atleast one of u and v is nonzero. Further, using the invariance of X_0 and $X_0^\perp$ and the fact that $X_0 \cap X_0^\perp = \{0\}$,

$$Ax = \lambda x \iff A_1 u = \lambda u \text{ and } A_2 v = \lambda v.$$

Thus, it follows that

$$\sigma_{\text{eig}}(A) = \sigma_{\text{eig}}(A_1) \cup \sigma_{\text{eig}}(A_2).$$

This completes the proof. ∎

Lemma 4.4.3 *Let A be a self adjoint operator on a Hilbert space X and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A. Let*

$$X_0 = N(A - \lambda_1 I) + \ldots + N(A - \lambda_k I)$$

and let $A_1 := A_{|X_0}$ and $A_2 := A_{|X_0^\perp}$. Then

(i) $\sigma_{\text{eig}}(A_1) = \{\lambda_1, \ldots, \lambda_k\}$,

(ii) $\sigma_{\text{eig}}(A_2) = \sigma_{\text{eig}}(A) \setminus \{\lambda_1, \ldots, \lambda_k\}$.

Proof. (i) It can be easily seen that $\{\lambda_1, \ldots, \lambda_k\} \subseteq \sigma_{\text{eig}}(A_1)$. Now, let $\lambda \in \sigma_{\text{eig}}(A_1)$. Then there exists a nonzero $x \in X_0$ such that $Ax = \lambda x$. Let $x_i \in N(A - \lambda_i I)$ for $i = 1, \ldots, k$ such that

$$x = x_1 + \ldots + x_k.$$

Since $N(A - \lambda_i I) \perp N(A - \lambda_j I)$ for $i \neq j$, we have

$$\|x\|^2 = \|x_1\|^2 + \ldots + \|x_k\|^2.$$

Hence, $x_i \neq 0$ for some $i \in \{1, \ldots, k\}$. Also, since $Ax = \lambda x$ and

$$\begin{aligned} Ax - \lambda x &= (Ax_1 - \lambda x_1) + \ldots + (Ax_k - \lambda x_k) \\ &= (\lambda_1 - \lambda)x_1 + \ldots + (\lambda_k - \lambda)x_k, \end{aligned}$$

it follows that $\lambda = \lambda_i \in \{\lambda_1, \ldots, \lambda_k\}$.

(ii) Let $\lambda \in \sigma_{\text{eig}}(A) \setminus \{\lambda_1, \ldots, \lambda_k\}$. By Lemma 4.4.2, we know that $\sigma_{\text{eig}}(A) = \sigma_{\text{eig}}(A_1) \cup \sigma_{\text{eig}}(A_2)$. Hence, by part (i), we obtain $\lambda \in \sigma(A_2)$. Next, suppose that $\lambda \in \sigma_{\text{eig}}(A_2)$. Then there exists a nonzero $x \in X_0^\perp$ such that $Ax = \lambda x$. Then, $\lambda \notin \{\lambda_1, \ldots, \lambda_k\}$, for if $\lambda = \lambda_i$ for some $i \in \{1, \ldots, k\}$, then we would have $Ax = \lambda_i x$ so that $x \in N(A - \lambda_i I) \subseteq X_0$, which would contradict the fact that $x \neq 0$. Thus, we have proved that $\lambda \in \sigma_{\text{eig}}(A) \setminus \{\lambda_1, \ldots, \lambda_k\}$ if and only if $\lambda \in \sigma_{\text{eig}}(A_2)$. ∎

Now, we state and prove the main theorem of this book, the so called *spectral theorem for a compact self adjoint operator.*

Theorem 4.4.4 *Let X be a Hilbert space and $A : X \to X$ be a nonzero compact self adjoint operator. Then*

$$A = \sum_{i \in \Lambda} \lambda_i P_i,$$

where $\{\lambda_j : j \in \Lambda\}$ is a countable set of real numbers which are the eigenvalues of A and, for each $i \in \Lambda$, P_i is the orthogonal projection onto the eigen space $N(A - \lambda_i I)$.

Proof. We know that the eigenspectrum of A is a countable set, say $\sigma_{\rm eig}(A) = \{\lambda_i : i \in \Lambda\}$, where $\Lambda = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$ if $\sigma_{\rm eig}(A)$ is a finite set and $\Lambda = \mathbb{N}$ if $\sigma_{\rm eig}(A)$ is an infinite set.

Case (i): *$\sigma_{\rm eig}(A)$ is a finite set.*

Suppose $\sigma_{\rm eig}(A) = \{\lambda_1, \ldots, \lambda_k\}$, where $\lambda_1, \ldots, \lambda_k$ are distinct. We know that each $N(A - \lambda_i I)$ is finite dimensional (cf. Theorem 4.1.8). Let $\{v_{ij} : j = 1, \ldots, n_i\}$ be an orthonormal basis of $N(A-\lambda_i I)$ for $i = 1, \ldots, k$. By Theorem 4.3.5), $N(A - \lambda_i I) \perp N(A - \lambda_j I)$ for $i \neq j$. Hence,

$$\bigcup_{i=1}^{k} \{v_{ij} : j = 1, \ldots, n_i\}$$

is an orthonormal basis of

$$X_k := N(A - \lambda_1 I) + \cdots + N(A - \lambda_k I).$$

By projection theorem, every $x \in X$ can be written uniquely as

$$x = u + v \quad \text{with} \quad u \in X_k,\, v \in X_k^{\perp}.$$

Note that

$$\langle x, v_{ij}\rangle = \langle u, v_{ij}\rangle \quad \forall\, i = 1, \ldots, k,\, j = 1, \ldots, n_i,$$

$$u = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \langle u, v_{ij}\rangle v_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \langle x, v_{ij}\rangle v_{ij}$$

so that

$$Ax = Au + Av = \sum_{i=1}^{k} \lambda_i P_i x + Av,$$

where P_i is the orthogonal projection onto $N(A - \lambda_i)$, i.e.,

$$P_i x = \sum_{j=1}^{n_i} \langle x, v_{ij}\rangle v_{ij}, \quad x \in X.$$

Hence, it is enough to prove that $Av = 0$. Since X_k is invariant under A, by Theorem 4.4.1, $X_k^{\perp}$ is also invariant under A. Therefore, $A_k := A_{|_{X_k^{\perp}}}$ is a compact self adjoint operator on $X_k^{\perp}$. We, in fact, show that $A_k = 0$, which would imply that $Av = 0$.

Suppose $A_k \neq 0$. Then, by Corollary 4.3.10, A_k will have a nonzero eigenvalue, say λ, which would be an eigenvalue of A as well. Let $y \in X_k^\perp$ be a corresponding eigenvector. Then we have $Ay = \lambda y$ so that the $\lambda = \lambda_i$ for some $i = 1, \ldots, k$, and hence $y \in N(A - \lambda_i I) \subseteq X_k$. This is a contradiction to the fact that $y \neq 0$.

Case (ii): $\sigma_{\text{eig}}(A)$ *is an infinite set.*

Suppose that $\lambda_1, \lambda_2, \ldots$ are the distinct eigenvalues of A. Without loss of generality assume that $|\lambda_i| \geq |\lambda_{i+1}|$ for every $i \in \mathbb{N}$. As earlier, let

$$X_k := N(A - \lambda_1 I) + \cdots + N(A - \lambda_k I).$$

Let $x \in X$. Again, by projection theorem,

$$x = u + v \quad \text{with} \quad u \in X_k,\, v \in X_k^\perp$$

and

$$Ax = \sum_{i=1}^{k} \lambda_i P_i x + Av,$$

where $\|v\| \leq \|x\|$. Hence,

$$\|Ax - \sum_{i=1}^{k} \lambda_i P_i x\| \leq \|A_k\|\, \|x\|,$$

where $A_k := A_{|_{X_k^\perp}}$ is a compact self adjoint operator on $X_k^\perp$. By Lemma 4.4.3,

$$\sigma_{\text{eig}}(A_k) = \sigma_{\text{eig}}(A) \setminus \{\lambda_1, \ldots, \lambda_k\}.$$

Hence, $\|A_k\| = |\lambda_{k+1}|$ so that

$$\|Ax - \sum_{i=1}^{k} \lambda_i P_i x\| \leq \|A_k\|\, \|x\| = |\lambda_{k+1}|\, \|x\|$$

and hence,

$$\|A - \sum_{i=1}^{k} \lambda_i P_i\| \leq |\lambda_{k+1}| \quad \forall\, k \in \mathbb{N}.$$

Since $\lambda_{k+1} \to 0$ as $k \to \infty$ (cf. Theorem 4.1.8), we obtain

$$A = \sum_{i=1}^{\infty} \lambda_i P_i.$$

This completes the proof. ∎

Corollary 4.4.5 *Let A be a compact self adjoint operator. Then there exists a sequence (B_n) of self adjoint finite rank operators such that*

$$\|A - B_n\| \to 0 \quad as \quad n \to \infty.$$

Proof. Follows from Theorem 4.4.4. ∎

Recall that in obtaining Theorem 4.4.4, one of the primary fact about a nonzero compact self adjoint operator that we used is that it has a nonzero eigenvalue. We know that this fact need not be true for a non-self adjoint compact operator (cf. Exmaple 4.1.5). However, we do have a representation similar to the one as in Theorem 4.4.4 for any compact operator on a Hilbert space, in terms of the so called *singular values.*

Theorem 4.4.6 (Singular value representation) *Let X and Y Hilbert spaces and $T : X \to Y$ be a compact operator. Then there exist a orthonormal basis $\{u_n : n \in \Lambda\}$ for $N(T)^\perp$, an orthonormal basis $\{v_n : n \in \Lambda\}$ for $\overline{R(T)}$ and $\{s_n : n \in \Lambda\} \subseteq [0, \infty)$ such that*

$$Tx = \sum_{n \in \Lambda} s_n \langle x, u_n \rangle v_n \qquad \forall\, x \in X,$$

where $\Lambda = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$ if $k = \operatorname{rank}(T) < \infty$, and $\Lambda = \mathbb{N}$ if $\operatorname{rank}(T) = \infty$.

Proof. Note that T^*T is a compact self adjoint operator so that by Theorem 4.4.4, there exists an orthonormal set $\{u_n : n \in \Lambda\}$ in X and $\{m_n : n \in \Lambda\} \subseteq [0, \infty)$

$$T^*Tx = \sum_{n \in \Lambda} \mu_n \langle x, u_n \rangle u_n, \quad x \in X,$$

where $\Lambda = \{1, \ldots, k\}$ for some $k \in \mathbb{N}$ or if $k = \operatorname{rank}(T) < \infty$, and $\Lambda = \mathbb{N}$ if $\operatorname{rank}(T) = \infty$. Also, we know that, if $\Lambda = \mathbb{N}$, then $\mu_n \to 0$ as $n \to \infty$. Note that

$$T^*Tu_n = \mu_n u_n \quad \forall\, n \in \Lambda.$$

Hence,

$$\mu_n = \langle \mu_n u_n, u_n \rangle = \langle T^*Tu_n, u_n \rangle = \langle Tu_n, Tu_n \rangle = \|Tu_n\|^2 \geq 0.$$

Taking

$$s_n = \sqrt{\mu_n} \quad \text{and} \quad v_n = \frac{Tu}{s_n} \quad \forall\, n \in \Lambda,$$

we have

$$Tu_n = s_n v_n \quad \text{and} \quad T^* v_n = s_n u_n \quad \forall\, n \in \Lambda.$$

We also know that

$$\langle x, u_n \rangle = 0 \quad \forall\, n \in \Lambda \implies T^*Tx = 0 \implies x \in N(T^*T) = N(T)$$

so that $\{u_n : n \in \Lambda\}$ is an orthonormal basis of $N(T)^\perp$. Also, for every $x \in X$,

$$\langle Tx, v_n \rangle = \langle x, T^* v_n \rangle = \langle x, s_n u_n \rangle = s_n \langle x, u_n \rangle$$

so that

$$\begin{aligned} \langle Tx, v_n \rangle = 0 \quad \forall\, n \in \Lambda \quad &\implies \quad \langle x, u_n \rangle = 0 \quad \forall\, n \in \mathbb{N} \\ &\implies \quad x \in N(T) \implies Tx = 0. \end{aligned}$$

Hence, $\{v_n : n \in \Lambda\}$ is an orthonormal basis of $R(T)$. Therefore, for every $x \in X$,

$$Tx = \sum_{n \in \mathbb{N}} \langle Tx, v_n \rangle v_n = \sum_{n \in \Lambda} \langle x, T^* v_n \rangle v_n = \sum_{n \in \Lambda} s_n \langle x, u_n \rangle v_n.$$

This completes the proof. ∎

Corollary 4.4.7 *Let X and Y Hilbert spaces and $T : X \to Y$ be a compact operator of infinite rank. Then there exists sequence of finite rank bounded operators T_n such that $\|T - T_n\| \to 0$ as $n \to \infty$. In fact,*

$$T_n x := \sum_{j=1}^{n} s_j \langle x, u_j \rangle v_j \qquad \forall\, x \in X,$$

where $\{(s_n, u_n, v_n) : n \in \mathbb{N}\}$ is as in Theorem 4.4.6, and

$$\|T - T_n\| \le \sup_{j > n} s_j.$$

If s_n, u_n, v_n are as in Theorem 4.4.6, then we have

$$Tu_n = s_n v_n, \qquad T^* v_n = s_n u_n.$$

Definition 4.4.2 Let $T \in \mathcal{K}(X)$, where X is a Hilbert space. The set

$$\{(s_n, u_n, v_n) : n \in \Lambda\}$$

obtained as in Theorem 4.4.6 is called a **singular system** for A. The numbers s_n are called the **singular values** of A with corresponding **singular vectors** u_n, v_n for $n \in \Lambda$. $\Diamond$

Let X be a Hilbert space and $T \in \mathcal{K}(X)$ be of finite rank, say $\operatorname{rank}(T) = k$. Then, by Theorem 4.4.6,

$$Tx = \sum_{i=1}^{k} s_n \langle x, u_n \rangle v_n, \quad x \in X,$$

where $\{(s_n, u_n, v_n) : n = 1, \ldots, k\}$ a singular system for A. Let us consider the operators

$$U : X \to \mathbb{K}^k, \quad B : \mathbb{K}^k \to \mathbb{K}^k, \quad V : \mathbb{K}^k \to Y,$$

defined by

$$Ux = \sum_{i=1} \langle x, u_i \rangle e_i,$$

$$B\left(\sum_{i=1}^{k} \alpha_i e_i\right) = \sum_{i=1}^{k} \alpha_i s_i e_i,$$

$$V\left(\sum_{i=1}^{k} \alpha_i e_i\right) = \sum_{i=1}^{k} \alpha_i v_i.$$

Then we have

$$T = VBU.$$

If $X = \mathbb{K}^n$ and $Y = \mathbb{K}^m$, then we see that U, B, V are the matrices

$$U = [\underline{u}_1\, \underline{u}_2\, \cdots\, \underline{u}_k], \quad B = \operatorname{diag}(s_1, \ldots, s_k), \quad V = [\underline{v}_1\, \underline{v}_2\, \cdots\, \underline{v}_k]^*.$$

In this special case, the representation $T = VBU$ is called the **singular value decomposition** of T.

Remark 4.4.1 Singular value decomposition of operators is effectively used in the solution of ill-posed operator equations which are mathematical formulations of many of the practically important inverse problems (cf. Nair [6]). $\Diamond$

4.5 Problems

1. Prove Theorem 4.1.2.

2. In Example 4.1.2. Let $X = c_{00}$ with $\|\cdot\|_p$ and $\lambda_n = 1/n$, $n \in \mathbb{N}$. Show that A is bijective, but $0 \in \sigma_{\mathrm{app}}(A)$.

3. In Example 4.2.4, show that $\sigma_{\mathrm{app}}(A) = [a, b]$.

4. Let $t_1, \ldots, t_n$ be distinct points in $[a, b]$. Construct $u \in C[a, b]$ such that if A is the operator as in Example 4.2.4, then $\sigma_{\mathrm{eig}}(A) = \{t_1, \ldots, t_n\}$.

5. Let (λ_n) be a bounded sequence of scalars and for $1 \le p \le \infty$, let $A : \ell^p \to \ell^p$ be defined by

$$(Ax)(j) = \lambda_j x(j), \qquad j \in \mathbb{N}, \quad x \in \ell^p.$$

Prove that

$$\sigma_{\mathrm{eig}}(A) = \{\lambda_n : n \in \mathbb{N}\} \quad \text{and} \quad \sigma(A) = \mathrm{cl}\,\{\lambda_n : n \in \mathbb{N}\}.$$

6. For $1 \le p \le \infty$, let A be the right shift operator on ℓ^p, that is,

$$Ax = (0, x(1), x(2), \ldots), \qquad x := (x(1), x(2), \ldots) \in \ell^p.$$

Prove that

$$\sigma_{\mathrm{eig}}(A) = \varnothing \quad \text{and} \quad \sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\}.$$

7. For $1 \le p \le \infty$, let A be the right shift operator on ℓ^p, that is,

$$Ax = (0, x(1), x(2), x(3), \ldots), \qquad x := (x(1), x(2), \ldots) \in \ell^p.$$

Prove that

$$\sigma_{\mathrm{eig}}(A) = \{\lambda \in \mathbb{K} : |\lambda| < 1\} \quad \text{and} \quad \sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\}.$$

8. For $1 \le p \le \infty$, let A be the left shift operator on ℓ^p, that is,

$$Ax = (x(2), x(3), \ldots), \qquad x := (x(1), x(2), \ldots) \in \ell^p.$$

Prove that

$$\sigma_{\mathrm{eig}}(A) = \{\lambda \in \mathbb{K} : |\lambda| < 1\} \quad \text{and} \quad \sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\}.$$

9. Give an example in each of the following:

   (a) An operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\sigma(A) = \varnothing$.

   (b) An operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $r_\sigma(A) < \|A\|$.

10. For $1 \le p \le \infty$, let $X = \ell^p$ and $A : \ell^p \to \ell^p$ be defined by

$$(Ax)(j) = \frac{x(j)}{j}, \qquad j \in \mathbb{N}, \quad x \in \ell^p.$$

   What are $\sigma_{\text{eig}}(A)$, $\sigma_{\text{app}}(A)$ and $\sigma(A)$? Why?

11. Let $X = C[a,b]$ with $\|\cdot\|_\infty$ and let $u \in C[a,b]$. Let $A : X \to X$ be defined by

$$(Ax)(t) = u(t)x(t), \qquad t \in [a,b], \quad x \in X.$$

   Prove that $\sigma_{\text{app}}(A) = \text{cl}\,\{u(t) : t \in [a,b]\}$.

12. Let $A \in \mathcal{B}(X)$, where X is a Hilbert space. Prove that $\lambda \in \sigma_{\text{eig}}(A)$ if and only if $R(A^* - \bar{\lambda}I)$ not dense.

13. Let $A \in \mathcal{B}(X)$, where X is a Hilbert space, and let $\sigma(A) \neq \varnothing$ and $\mu \in \mathbb{K}$. Prove the following:

   (a) $\sigma(A - \mu I) = \{\lambda - \mu : \lambda \in \sigma(A)\}$.

   (b) If $\mu \in \rho(A)$, then $\sigma((A - \mu I)^{-1}) = \left\{\dfrac{1}{\lambda - \mu} : \lambda \in \sigma(A)\right\}$.

   (c) If A is a normal operator and $\mu \in \rho(A)$, then

$$r_\sigma((A - \mu I)^{-1}) = \frac{1}{\text{dist}\,(\mu, \sigma(A))}.$$

14. Let A be a compact operator on a Hilbert space and $0 \neq \lambda \in \mathbb{K}$. Without using Riesz lemma prove that $N(A - \lambda I)$ is finite dimensional.

15. For $A \in \mathcal{B}(X)$, prove that $\|A\|^2 = r_\sigma(A^*A)$.